Jesse Gerald Chaney

ON THE GENERALIZED CIRCUIT THEORY AS APPLIED TO ANTENNAS AND RADIATING LINES.

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Research Paper No. 1

U. S. Naval Academy Press—4-5-51—130

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12	8	$\bar{J}_0$ should be $\bar{J}_0 f(P_1)$
18	6	$\bar{A}_1[\quad] \cdot d\bar{s}_2$ should be $\bar{A}_1[\quad] \cdot d\bar{s}_1$
24	3	2,3(7) should be 2.3(6)
24	5,6	subscripts 21 should be 12
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67	25	in last line, all subscripts should be 2
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CONTENTS

INTRODUCTION

1.1 Within recent years, much interest has been shown in the mathematical theory for finding the input impedance of various types of antenna configurations. There have been quite a number of different theories, or methods, advanced for finding both the self impedance of a wire antenna and the mutual impedance of two such antennas.

Since wire or tubular antennas are usually operated near resonance such that the currents to a fair degree of approximation may be assumed to be distributed sinusoidally from the ends([1,2,3,4,5,10,11,39,41]), provided the rasius a satisfies the inequalities

$$ka \ll 1, \quad a \ll l, \quad k = \frac{2\pi}{\lambda}, \tag{1}$$

$2l$ being the length of the antenna, one of the older methods for deriving formulas for antenna impedances postulated such a current distribution. However, as the radius becomes quite appreciable with respect to the wave length corresponding to the frequency of operation, the current distribution departs considerably from the sinusoidal. Thus, theories arose in which the *a priori* current distribution was not assumed to be known.

One of the better known methods was proposed by S. A. Schelkunoff, who considered the solution of Maxwell's electromagnetic field equations and matched the fields at the boundaries([1,17,42,44,71,78]). In this manner, he arrived at a terminal admittance which could be evaluated by a series of fractional order Bessel functions. Since the series converged rather slowly, he approximated the input impedance of a hollow cylindrical antenna by finding an inverse terminal impedance by the classical method for a vanishingly thin bi-conical antenna

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whose current could be legitimately considered sinusoidal and then used non-uniform transmission line theory for finding the impedance([1]).

Schelkunoff's method has been slightly modified by J. R. Whinnery and associates to include effects of the input configuration upon the impedance([61,62]).

In 1930-38, E. Hallen published his now famous papers([13,75]) in which he solved for the input impedance of a cylindrical antenna by assuming a continuity of the vector potential across the slice generator and a discontinuity of the scalar potential across the slice, resulting in an integral equation which he solved by successive approximations, expanding in terms of the parameter

$$\Omega = 2\ln\frac{2l}{a} \; . \tag{2}$$

In 1941, J. A. Stratton and L. J. Chu considered the problem from the point of view of forced oscillations([72,73,74]). Their solutions were arrived at by matching the fields at the boundaries.

The interpretation of Hallen's method by others led to the general opinion that certain modifications were necessary to bring the theoretical results into agreement with measured results([9,16,18,19,41,44,45,46,76]). However, Hallen has shown recently that none of the published curves attributed to him were actually in accordance with his theory([24,28,29]), and that his theory does check with measurements.

Hallen also shows that the current wave travels with a variable phase velocity, speeding up as the end of the wire is approached([24]). Thus, the attenuation is not exponential. In fact, he contends that it is not at all proper to consider an inductance and capacitance per unit length for an antenna.

It is apparent that there is still some difference of opinion as to the better method of finding the self impedance of an antenna.

1.2 A single antenna radiates energy in directions other than that desired in point to point comunication service. Thus means of increasing the directivity were sought.

During 1927-33, S. Uda[67], H. Yagi[64], and K. Tani[69] investigated the possibility of using parasitic radiators for increasing the directivity and gain in a desired direction. Such a parasitic array has become known as a Yagi array.

Many measurements have been published for the radiation resistance and mutual impedance of beam antennas[55,56,63]. Also, much literature has appeared on the problem of determining the mutual impedances theoretically[1,4,6,7,8,9,11,12,31,35,37,47].

One of the first methods investigated was that of P. S. Carter[12]. He used what has become known as the induced e.m.f. method, assuming sinusoidally distributed currents. However, Carter considered only half-wave antennas. S. A. Schelkunoff and A. Aharoni[1,4] have extended Carter's method to two antennas of equal length but not restricted to half-wave antennas.

Charles W. Harrison, Jr.[6,7,8,9,10,11] has considered numerous array configurations by means of an integral equation method due to Ronold W. P. King [11,27,31] and himself. This method is essentially a modification of Hallen's method in which the antennas are assumed to be driven by slice generators, firstly in phase, and secondly in phase opposition.

Although he states that his method is possible of extension to any number of coplanar antennas of arbitrary lengths, in the case of

a Yagi array, Harrison actually carries through the solution only for two elements. Since this method does not assume an a priori current distribution, the computation of the mutual impedance of a pair of elements in fixed position varies with the addition of new elements. In other words, as the number of elements is increased, the entire solution of the system of integral equations must be carried out by the method of successive approximations. Thus one immediately discards the method if it is desired to compute the input impedance of a multi-element parasitic array.

Furthermore, experimental measurements by W. E. Stoney[55] of the mutual impedance of a pair of half-wave antennas at various spacings checked with computations by Carter's method much more closely than computations by Harrison's method[6,12]. Also, measured values made upon a three element beam were found by Stoney[56] to check fairly well with values computed by Carter's circuit method. However, Stoney stated that he did not have available a formula for the mutual impedance of two parallel elements each normal to the line of centers and of unequal length as required for the solution of a tuned Yagi array, and hence he maintained all elements of the same length.

1.3 After searching the literature for a method whereby an engineer could solve for the approximate input impedance of a Yagi array, it was decided to fall back upon the conventional a priori sinusoidally distributed currents. This assumption is a good approximation to the actual current distribution as long as inequalities 1.1(1) are satisfied and provided the lengths are in the neighborhood of a half-wave. This is partially verified by the agreement between measured and computed

results for a three element, horizontally oriented, Yagi array above an imperfect ground.

Also, after carrying through the derivations herein, a British report[63] edited by B. Starnecki and E. Fitch was located which listed a formula for the mutual impedance of two parallel elements of unequal length with each normal to the line of centers, and which gave a comparison of values computed by this formula with experimental results. Again the comparison was quite favorable. Although no derivation of the formula was given, it was stated that the formula was derived by J. H. Tait by Aharoni's method of extending Carter's circuit theory for assumed sinusoidal currents[4]. This formula is identical with one which appears herein as a special case of the formula derived for the mutual impedance of two parallel-staggered antennas of unequal length, and which is applied in a numerical solution.

1.4 It is now proposed to derive a formula for the mutual impedance of two straight cylindrical antennas of unequal length and of an arbitrary orientation. The antennas will be assumed to be of half-lengths l_1 and l_2, to have radii a_1 and a_2, and to satisfy the inequalities

$$ka_1 \ll 1, \; ka_2 \ll 1, \; a_1 \ll l_1, \; a_2 \ll l_2 \; . \tag{1}$$

The problem will be formulated as an extension of fundamental circuit theory, that is, starting with Ohm's law for fields, the generalized field circuit will be developed for two meshes, and then, postulating sinusoidally distributed currents, double line integrals will be derived for the mutual impedances.

The double line integral expressions will be generalized to give

the mutual impedance of two wires carrying currents whose phase relations are functions of position. It will be shown that the generalized mutual impedances are symmetric in the subscripts.

1.5 In carrying out the integrations for the standing wave case, the most general case that can be integrated by rigorous methods is that for two parallel-staggered antennas of unequal lengths. However, the more general formula is obtained by an approximation method analogous with curve fitting which is believed to yield quite good results.

The rigorously integrated parallel-staggered case will yield both the parrallel case and the collinear case, the former being the one desired for the Yagi array. The formula for the parallel case checks with published results both for antennas of unequal length[(63)] and for antennas of equal length[(1)]. The collinear case will be used in indicating a procedure for solving for the input impedance of a Yagi array operated on its harmonic frequencies.

Also it will be shown how one should consider the effects of an imperfect ground upon the input impedance. As an illustration, the theory will be checked against the numerical solution of a three element Yagi array designed for operation on 14.28 megacycles per second, taking into consideration the finite conductivity and dielectric constant of the local earth, and the results will be compared with measured values. The comparison proves to be gratifying. For consistency, the self impedances will be found by the conventional method employed herein.

It should be pointed out that although comparisons have been made

by previous investigators, they were made upon half-length vertical antennas immediately above an artificial highly conducting ground and thus did not simulate the actual operating conditions of a horizontal array.

1.6 Also, the generalized double line integral will be integrated to give the mutual impedance between two parallel wires of equal length, assumed to be carrying unattenuated travelling waves of current in phase opposition. Then, as in the case of standing waves, the mutual impedance formula will be applied, in the same manner customarily used in applying Neumann's formula for lumped inductances, for finding the intrinsic impedance of a single wire in free space assumed to be carrying a travelling wave of current. The resistive component of this impedance is identical with the radiation resistance of such a wire as derived by integration of the Poynting vector over a spherical surface surrounding the wire[2].

The results will be interpreted physically in terms of an open wire transmission line assumed to act as if it were terminated in its characteristic impedance. This yields an expression for the radiation impedance of an open wire transmission line, which expression vanishes in the limit as the line spacing tends to zero. This, of course, neglects the attenuation of the current and thus does not give the same expression for the radiation resistance of a transmission line as derived by J. E. Storer and R. W. P. King[21], who used a series expansion approximation in carrying out their integrations.

Also, the results will be used in deriving an expression for the approximate input impedance of a Beverage[22] wave antenna.

1.7 As in the case of standing waves, the most general case that can be integrated rigorously for the mutual impedance of two wires carrying unattenuated travelling waves of current is that for parallel-staggered wires. These results will be interpreted in such a manner as to give the coupling and radiation impedance of parallel transmission lines within the same vicinity, each line being terminated in its characteristic impedance.

1.8 Due to the generality of the problems involved, vector geometry will be used throughout the developments.

1.9 The following symbols are used in the remaining portion of the text:

$2\bar{l}_1$ = vector axis of antenna one

$2\bar{l}_2$ = vector axis of antenna two

$\bar{a}_i$ = unit vector

$\bar{r}_{12}$ = vector from any point on axis of antenna one to any point axis of antenna two

$\bar{r}_c$ = vector from centroid of antenna one to centroid of antenna two

$\bar{\Delta}$ = vector operator del

$\bar{\bar{\Delta}}_i$ = vector operator deltil which adds k^2 times a vector to the gradient of the divergence of the vector

$k = \frac{2\pi}{\lambda} = \omega\sqrt{\mu_o\varepsilon_o}$ = wave number

$R_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi$ ohms = intrinsic impedance of free space

$\mu_o = 4\pi(10)^{-7}$ henries per meter = permeability of free space

$\varepsilon_o = \frac{1}{36\pi}(10)^{-9}$ farads per meter = permitivity of free space

$\bar{A}$ = retarded vector potential

σ = conductivity in mhos per meter

$\bar{I}$ = current density in amperes per square meter

$\bar{J}$ = current density in amperes per meter

$\bar{I}$ = current in amperes

I' = the complex conjugate of I

δ = penetration depth of current

R_s = surface resistivity

$\bar{E}$ = electric field strength in volts per meter

$\bar{s}_1$ = vector from the centroid of antenna one to any point on the positive axis of antenna one

$\bar{s}_2$ = vector from centroid of antenna two to any point on the positive axis of antenna two

P_{11} = negative terminus of vector axis of antenna one

P_{10} = centroid of vector axis of antenna one

P_{12} = positive terminus of vector axis of antenna one

P_{21} = negative terminus of vector axis of antenna two

P_{20} = centroid of vector axis of antenna two

P_{22} = positive terminus of vector axis of antenna two

r'_{21} = distance from P_{21} to any point on vector axis of antenna one

r'_{20} = distance from P_{20} to any point on vector axis of antenna one

r'_{22} = distance from P_{22} to any point on vector axis of antenna one

h_1 = the projection of the line of centers vector upon antenna one

z_{2i} = projection of one of the nine diagonal lengths upon antenna one

y_{2i} = value of z_{2i} for parallel antennas

r'_i = one of the nine diagonal lengths between the points P_{ij}

$\bar{L}^2_{i1} = r'_i - z_{2i}$

$\bar{\bar{L}}^2_{i2} = r'_i + z_{2i}$

$L_{ij} = \bar{\bar{L}}^2_{ij} = \bar{\bar{L}}^1_{ij}$ for parallel antennas

ρ = Greek letter rho

Z_{12} = impedance reflected into antenna one by the current in antenna two

Z'_{12}, Z''_{12} = portions of Z_{12}

(r, θ, ϕ) = spherical coordinates

(ρ, ϕ, z) = cylindrical coordinates

Ci = cosine integral function

Si = sine integral function

C = Euler's constant

II

THE GENERALIZED CIRCUIT

2.1 Ohm's law for fields is[5]

$$\bar{E} = \frac{1}{\sigma}\bar{i} \qquad (1)$$

in which

$$\bar{E} = \bar{E}_o + \bar{E}_i \, , \quad \bar{i} = \bar{i}_o f(P).$$

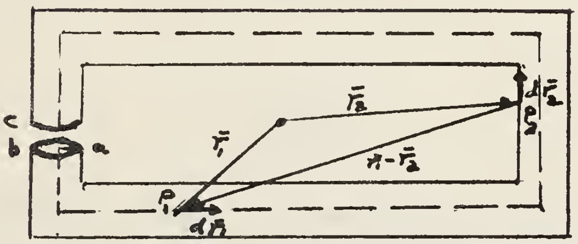


Figure 1

Now, $\bar{E}_o$ is the applied field and hence the line integral

$$\oint \bar{E}_o \cdot d\bar{r}$$

will be assumed to be independent of the path of integration, whereas, $\bar{E}_i$ is the induced field or that portion of the total field which arises from the current densities within the circuit and is given by[5]

$$\bar{E}_i = \frac{1}{j\omega\varepsilon_o}\bar{\Delta}_1(\bar{\Delta}_1 \cdot \bar{A}) - j\omega\mu_o\bar{A} \qquad (2)$$

with

$$\bar{A} = \int_V \frac{\bar{i}e^{-jkr_{21}}}{4\pi r_{21}}\, dv \qquad (3)$$

in which

$$r_{21} = |\bar{r}_1 - \bar{r}_2| \cdot$$

Thus, assuming $f(P) = 1$ for P within (c b)

$= f(P_1)$ for P within (b c)

the circuit equation becomes[5]

$$\bar{E}_o = \frac{1}{\sigma}\bar{i}_o f(P_1) + j\omega\mu_o\bar{A} - \frac{1}{j\omega\varepsilon_o}\bar{\Delta}_1(\bar{\Delta}_1 \cdot \bar{A}) \qquad (4)$$

the subscript upon the vector operator, del, indicating the differentiation to be at point P_1 .

2.2 At high frequencies the current densities may be written as surface current densities[1,3,5], that is

$$\bar{J} = \frac{\delta \bar{i}_s}{(1 + j)}$$

in which

$$\delta = \sqrt{\frac{\pi f \mu}{\sigma}}$$

is known as the penetration depth. Equation 2.1(4) may then be written

$$\bar{E}_0 = \frac{(1+j)}{\delta\sigma}\bar{J}_0 + j\frac{R_0}{k}[\bar{\Delta}_1\bar{\Delta}_1\cdot + k^2]\bar{A} \qquad (1)$$

with

$$\bar{A} = \int_s \frac{\bar{J}\, e^{-jkr_{12}}}{4\pi r_{12}} dS$$

and R_0 the intrinsic impedance of free space, and in which the surface of integration is that of the wire of radius a . If the current is sufficiently isolated such that the current may be assumed uniformly distributed around the surface of the wire, and if

$$ka \ll \lambda$$

it is permissible to write

$$\bar{I} = 2\pi a\bar{J}$$

and

$$\bar{A} = \oint_b^c \frac{\bar{I} e^{-jkr_{21}}}{4\pi\, r_{21}} dr_2 \quad . \qquad (2)$$

Thus, if R_s is the surface resistivity, and if the vector operator deltil is defined as

$$\bar{\bar{\Delta}} = [\bar{\Delta}\bar{\Delta}\cdot + k^2]$$

equation (1) may be written

$$\bar{E}_o = \frac{R_s(1+j)}{2\pi a}\bar{I}_o f(P_1) + j\frac{30}{k}I_o\oint_b^c \bar{\bar{\Delta}}_1[f(P_2)\frac{e^{-jkr_{21}}}{r_{21}}d\bar{r}_2] \ . \qquad (3)$$

Taking the scalar product with the complex conjugate of $\frac{1}{2}\bar{I}dr_1$,

$\frac{1}{2}\bar{E}_o\cdot\bar{I}_o' f(P_1)' dr_1 =$

$$\frac{R_s(1+j)}{4\pi a}|I_o f(P_1)|^2 dr_1 + j\frac{30|I_o|^2}{k}\oint_b^c f(P_2)f(P_1)'\bar{\bar{\Delta}}_1[\frac{e^{-jkr_{21}}}{r_{21}}d\bar{r}_2]\cdot d\bar{r}_1 \quad . \qquad (4)$$

Integrating (4) to obtain the power input, defining

$$V = -\oint_c^b \bar{E}_o\cdot d\bar{r}_1$$

as the applied voltage, and

$$R_1 = \frac{R_s}{2\pi a}\oint_b^c |f(P_1)|^2 d\bar{r}_1$$

as the ohmic resistance, and recalling that the applied field is assumed to be conservative, the input impedance defined as

$$Z_{in} = \frac{V}{I_o}$$

becomes

$$Z_{in} = Z_1 + j\frac{30}{k}\oint_b^c\oint_b^c f(P_2)f(P_1)'\bar{\bar{\Delta}}_1[\frac{e^{-jkr_{21}}}{r_{21}}d\bar{r}_2]\cdot d\bar{r}_1 \qquad (5)$$

in which

$$Z_1 = R_1(1+j)$$

is the internal impedance of the conductor.

2.3 Now consider two circuits with no direct coupling.

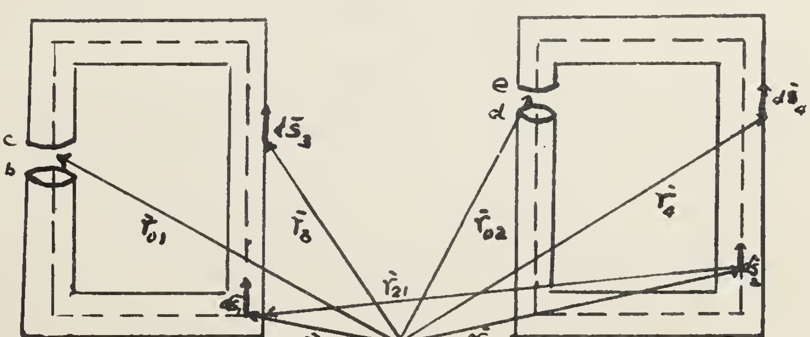


Figure 2

Let

$$\bar{s}_1 = \bar{r}_1 - \bar{r}_{01} \ , \ d\bar{s}_1 = d\bar{r}_1$$

$$\bar{s}_2 = \bar{r}_2 - \bar{r}_{02} \ , \ d\bar{s}_2 = d\bar{r}_2 \ , \text{ etc.}$$

Also, let

$$\bar{s}_1 = |\bar{s}_1|\bar{a}_1 = s_1\bar{a}_1$$

$$\bar{s}_2 = |\bar{s}_2|\bar{a}_2 = s_2\bar{a}_2 \ , \text{ etc.}$$

Following the same procedure as in 2.2, one now has

$$\bar{A}_1 = \oint_b^c \frac{\bar{I}_{01} f(P_3) e^{-jkr_{31}}}{4\pi r_{31}} d\bar{s}_3 + \oint_d^e \frac{\bar{I}_{02} f(P_2) e^{-jkr_{21}}}{4\pi r_{21}} d\bar{s}_2 \tag{1}$$

and

$$\bar{A}_2 = \oint_b^c \frac{\bar{I}_{01} f(P_1) e^{-jkr_{12}}}{4\pi r_{12}} d\bar{s}_1 + \oint_d^e \frac{\bar{I}_{02} f(P_4) e^{-jkr_{42}}}{4\pi r_{42}} d\bar{s}_4 \ . \tag{2}$$

After substituting into 2.2(3), integrating for the power inputs, and striking out $\frac{1}{2}I_{01}^*$ and $\frac{1}{2}I_{02}^*$, respectively, one obtains the circuit equations:

$$\begin{aligned} V_1 = {} & I_{01}\left\{\frac{j30}{k}\oint_b^c\oint_b^c f(P_1)' f(P_3)\bar{\bar{\Delta}}_1\left[\frac{e^{-jkr_{31}}}{r_{31}} d\bar{s}_3\right]\cdot d\bar{s}_1 + Z_1\right\} \\ & + I_{02}\left\{\frac{j30}{k}\oint_b^c\oint_d^e f(P_1)' f(P_2)\bar{\bar{\Delta}}_1\left[\frac{e^{-jkr_{21}}}{r_{21}} d\bar{s}_2\right]\cdot d\bar{s}_1\right\} \end{aligned} \tag{3}$$

$$\begin{aligned} V_2 = {} & I_{01}\left\{\frac{j30}{k}\oint_b^c\oint_d^e f(P_1) f(P_2)'\bar{\bar{\Delta}}_2\left[\frac{e^{-jkr_{12}}}{r_{12}} d\bar{s}_1\right]\cdot d\bar{s}_2\right\} \\ & + I_{02}\left\{\frac{j30}{k}\oint_d^e\oint_d^e f(P_4) f(P_2)'\bar{\bar{\Delta}}_2\left[\frac{e^{-jkr_{42}}}{r_{42}} d\bar{s}_4\right]\cdot d\bar{s}_2 + Z_2\right\} \end{aligned} \tag{4}$$

or

$$\begin{aligned} V_1 &= I_{01}Z_{11} + I_{02}Z_{12} \\ V_2 &= I_{01}Z_{21} + I_{02}Z_{22} \end{aligned} \tag{5}$$

Hence, the impedances may be written

$$Z_{12} = \frac{j30}{k}\oint_b^c\oint_d^e f(P_1)'f(P_2)\bar{\Delta}_1\left[\frac{e^{-jkr_{21}}}{r_{21}}d\bar{s}_2\right]\cdot d\bar{s}_1 \qquad (6)$$

$$Z_{21} = \frac{j30}{k}\oint_b^c\oint_d^e f(P_1)f(P_2)'\bar{\Delta}_2\left[\frac{e^{-jkr_{12}}}{r_{12}}d\bar{s}_1\right]\cdot d\bar{s}_2 \qquad (7)$$

$$Z_{11} = \frac{j30}{k}\oint_b^c\oint_b^c f(P_1)'f(P_3)\bar{\Delta}_1\left[\frac{e^{-jkr_{31}}}{r_{31}}d\bar{s}_3\right]\cdot d\bar{s}_1 + Z_1 \qquad (8)$$

$$Z_{22} = \frac{j30}{k}\oint_d^e\oint_d^e f(P_2)'f(P_4)\bar{\Delta}_2\left[\frac{e^{-jkr_{42}}}{r_{42}}d\bar{s}_4\right]\cdot d\bar{s}_2 + Z_2 \qquad (9)$$

It may be observed that as formulated above, the impedances are not symmetric in the subscripts unless the current distribution functions are real, that is, unless the currents are every where in time phase within the circuits. Also, it should be noted that the self impedances are the mutual impedances between the axes and surface elements, which is to be expected, since the paths of integration are chosen in the conventional manner to avoid the infinite reactance of a circuit of dimensionless cross section. It may further be observed that equations (6) to (9), inclusive, are the basic equations for the well known <u>induced</u> <u>e.m.f.</u> method for determining antenna impedances, although as usually formulated, the current functions are required to be real.

<u>2.4</u> The reciprocity theorem[1,3] implies that

$$Z_{21} = Z_{12} \; . \qquad (1)$$

However, this is not apparent from 2.3(6) and 2.3(7) for the more general cases, as the integrands are not necessarily equivalent. Perhaps a closer examination of the <u>graddiv</u> terms will lead to a better insight into this problem.

Consider the case of two wire antennas with $f(P_1)$ and $f(P_2)$ assumed to be real functions. Now construct a spherical coordinate system at each of the points P_1 and P_2, with the polar axes along $\bar{a}_2$ and with

$$0 \leq \cos^{-1}\bar{a}_1\cdot\bar{a}_2 \leq \frac{\pi}{2} \quad .$$

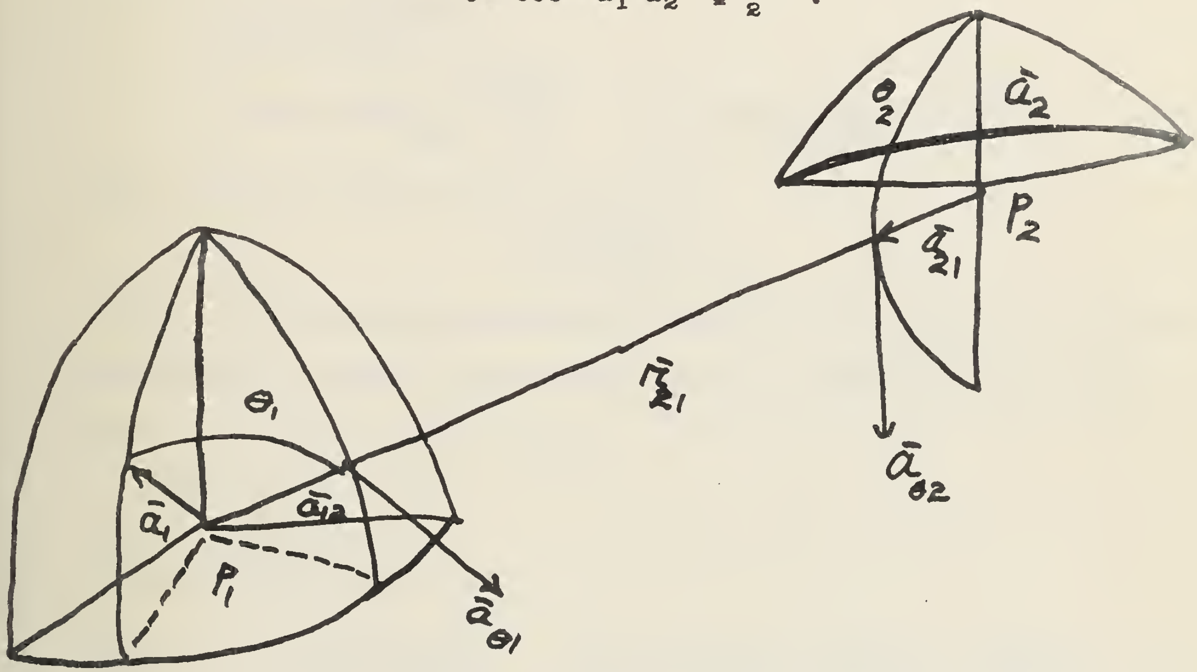


Figure 3.

Now, since

$$\bar{\Delta}_1\cdot\bar{a}_2 = \bar{\Delta}_1\cdot(\bar{a}_{21}\cos\theta_2 - \bar{a}_{\theta 2}\sin\theta_2) = 0 \ ,$$

$$\bar{\Delta}_1\bar{\Delta}_1\left(\bar{a}_2\frac{e^{-jkr_{21}}}{r_{21}}\right) = \bar{\Delta}_1\left[(\bar{a}_2\cdot\bar{a}_{21})\frac{\partial}{\partial r_{21}}\frac{e^{-jkr_{21}}}{r_{21}}\right]$$

$$= (\bar{a}_2\cdot\bar{a}_{21})\bar{\Delta}_1\frac{\partial}{\partial r_{21}}\frac{e^{-jkr_{21}}}{r_{21}} + \left(\frac{\partial}{\partial r_{21}}\frac{e^{-jkr_{21}}}{r_{21}}\right)\bar{\Delta}_1(\bar{a}_2\cdot\bar{a}_{21}) \ , \qquad (2)$$

Since

$$\bar{a}_2\cdot\bar{a}_{21} = \cos\theta_2$$

$$\bar{\Delta}_1(\bar{a}_2\cdot\bar{a}_{21}) = \bar{a}_{\theta 2}\frac{\partial}{r_{21}\partial\theta_2}\cos\theta_2 = -\bar{a}_{\theta 2}\frac{\sin\theta_2}{r_{21}} \quad . \qquad (3)$$

But

$$\bar{a}_2\cdot\bar{a}_{\theta 2} = - \sin\theta_2$$

hence

$$\bar{\Delta}_1\bar{\Delta}_1\cdot(\bar{a}_2\frac{e^{-jkr_{21}}}{r_{21}}) =$$

$$\bar{a}_{21}(\bar{a}_2\cdot\bar{a}_{21})\frac{\partial^2}{\partial r_{21}^2}\;\frac{e^{-jkr_{21}}}{r_{21}} + \bar{a}_{\theta 2}(\bar{a}_2\cdot\bar{a}_{\theta 2})\frac{\partial}{r_{21}\partial r_{21}}\;\frac{e^{-jkr_{21}}}{r_{21}}\quad , \qquad (4)$$

Thus, it may be observed that the vector

$$\bar{\bar{\Delta}}_1(\frac{e^{-jkr_{21}}}{r_{21}}\;d\bar{s}_2)$$

contains three coplanar components, one in the direction of the current element $d\bar{s}_2$, a second in the radial direction toward the element $d\bar{s}_1$, and a third normal to the second.

From (4),

$$\bar{a}_1\cdot(\bar{\Delta}_1\bar{\Delta}_1\cdot\frac{e^{-jkr_{21}}}{r_{21}}\bar{a}_2) =$$

$$(\bar{a}_1\cdot\bar{a}_{21})(\bar{a}_2\cdot\bar{a}_{21})\frac{\partial}{\partial r_{21}^2}\;\frac{e^{-jkr_{21}}}{r_{21}} + [(\bar{a}_1\cdot\bar{a}_{\theta 2})(\bar{a}_2\cdot\bar{a}_{\theta 2})\frac{\partial}{r_{21}\partial r_{21}}\;\frac{e^{-jkr_{21}}}{r_{21}}]\;,(5)$$

and similarly

$$\bar{a}_2\cdot(\bar{\Delta}_2\bar{\Delta}_2\cdot\bar{a}_1\frac{e^{-jkr_{12}}}{r_{12}}) =$$

$$(\bar{a}_2\cdot\bar{a}_{12})(\bar{a}_1\cdot\bar{a}_{12})\frac{\partial}{\partial r_{12}^2}\;\frac{e^{-jkr_{12}}}{r_{12}} + [(\bar{a}_2\cdot\bar{a}_{\theta 1})(\bar{a}_1\cdot\bar{a}_{\theta 1})\frac{\partial}{r_{12}\partial r_{12}}\;\frac{e^{-jkr_{12}}}{r_{12}}]. \qquad (6)$$

Subtracting (6) from (5) and using

$$\bar{r}_{21} = -\bar{r}_{12}$$

one obtains

$$\bar{a}_1\cdot[(\bar{\Delta}_1\bar{\Delta}_1\cdot\bar{a}_2\frac{e^{-jkr_{21}}}{r_{21}})] - \bar{a}_2\cdot[\bar{\Delta}_2\bar{\Delta}_2\cdot\bar{a}_1\frac{e^{-jkr_{12}}}{r_{12}}] = \begin{vmatrix}\bar{a}_1\cdot\bar{a}_{\theta 2} & \bar{a}_1\cdot\bar{a}_{\theta 1}\\ \bar{a}_2\cdot\bar{a}_{\theta 1} & \bar{a}_2\cdot\bar{a}_{\theta 2}\end{vmatrix}\frac{\partial e^{-jkr_{21}}}{r_{21}\partial r_{21}\;r_{21}}\;.\;(7)$$

To show that the determinant on the right in (7) vanishes identically, write

$$\bar{a}_1 = \bar{a}_{12}(\bar{a}_{12}\cdot\bar{a}_1) + \bar{a}_{\theta 1}(\bar{a}_{\theta 1}\cdot\bar{a}_1) \tag{8}$$

and

$$\bar{a}_2 = \bar{a}_{21}(\bar{a}_{21}\cdot\bar{a}_2) + \bar{a}_{\theta 2}(\bar{a}_{\theta 2}\cdot\bar{a}_2) \ , \tag{9}$$

form the scalar products $\bar{a}_1\cdot\bar{a}_2$ and $\bar{a}_2\cdot\bar{a}_1$, respectively, and subtract (8) from (9). Thus, it follows that

$$\bar{\bar{\Delta}}_1[\frac{e^{-jkr_{21}}}{r_{21}}d\bar{s}_2]\cdot d\bar{s}_2 = \bar{\bar{\Delta}}_2[\frac{e^{-jkr_{12}}}{r_{12}}d\bar{s}_1]\cdot d\bar{s}_2 \quad . \tag{10}$$

Substitution of (10) into 2.3(6) shows that equation (1) holds for the case under consideration.

Now, a closer examination of 2.1(6) shows that Z_{in} is dependent upon the order of the selection of the two paths of integration. However, neither current path is actually along the axis and one should certainly make no distinction between the two paths. To eliminate this anomaly, Z_{in} will be taken as the arithmetic mean of the values obtained by the two possible orders of path selections. Since

$$\tfrac{1}{2}[f(P_1)'f(P_2) + f(P_2)'f(P_1)] = Re[f(P_1)'f(P_2)] = Re[f(P_2)'f(P_1)] \ ,$$

in view of (10), the expression for Z_{in} becomes

$$Z_{in} = Z_1 + j\,\frac{30}{k}\oint_b^c\oint_b^c Re[f(P_1)'f(P_2)]\ \bar{\bar{\Delta}}_1[\frac{e^{-jkr_{21}}}{r_{21}}d\bar{r}_2]\cdot d\bar{r}_1. \tag{11}$$

The other formulas will be modified so that they conform with (11).

Thus:

$$Z_{12} = j\,\frac{30}{k}\oint_b^c\oint_d^e Re[f(P_1)'f(P_2)]\ \bar{\bar{\Delta}}_1[\frac{e^{-jkr_{21}}}{r_{21}}d\bar{s}_2]\cdot d\bar{s}_1 \tag{12}$$

$$Z_{21} = j\,\frac{30}{k}\oint_b^c\oint_d^e Re[f(P_2)'f(P_1)]\ \bar{\bar{\Delta}}_2[\frac{e^{-jkr_{12}}}{r_{12}}d\bar{s}_1]\cdot d\bar{s}_2 \tag{13}$$

$$Z_{11} = j\,\frac{30}{k}\oint_b^c\oint_b^c Re[f(P_1)'f(P_3)]\ \bar{\bar{\Delta}}_1[\frac{e^{-jkr_{31}}}{r_{31}}d\bar{s}_3]\cdot d\bar{s}_1 + Z_1 \tag{14}$$

$$Z_{22} = j\,\frac{30}{k}\oint_d^e\oint_d^e Re[f(P_2)'f(P_4)]\ \bar{\bar{\Delta}}_2[\frac{e^{-jkr_{42}}}{r_{42}}d\bar{s}_4]\cdot d\bar{s}_2 + Z_2 \tag{15}$$

The resulting formulas (12) to (16), inclusive, are perfectly symmetric in the subscripts and hence

$$Z_{21} = Z_{12}$$

holds for any two wire antennas even though the currents are not every where in time phase along the antennas.

III

MUTUAL IMPEDANCE OF OPEN WIRE ANTENNAS

3.1 In an attempt to determine the mutual impedance of any two symmetrically fed open wire antennas of arbitrary orientations, assume antenna one of length $2l_1$ and antenna two of length $2l_2$. Furthermore, let their directions be indicated by the unit vectors $\bar{a}_1$ and $\bar{a}_2$, respectively, In other words, antenna one is represented by the vector $2\bar{l}_1$ and antenna two by $2\bar{l}_2$. Let $\bar{r}_c$ be the vector from P_{20}, the mid-point of antenna two, to P_{10}, the mid-point of antenna one. Let $\bar{s}_2$ be the vector from P_{20} to any point within the interval $(P_{20}P_{22})$, and let $\bar{s}_1$ be from P_{10} to any point within the interval $(P_{10}P_{12})$. Let $\bar{r}_{21}$ be the vector from the terminus of $\bar{s}_2$ to the terminus of $\bar{s}_1$. Then

$$\bar{r}_{21} = \bar{r}_c + \bar{s}_1 - \bar{s}_2 \tag{1}$$

and

$$\bar{r}_{12} = -\bar{r}_{21} = -\bar{r}_c - \bar{s}_1 + \bar{s}_2 \,. \tag{2}$$

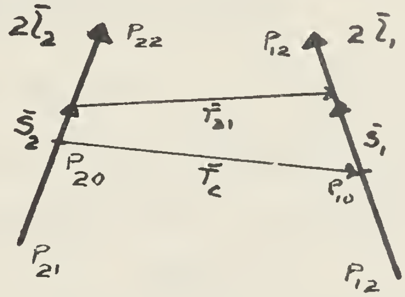


Figure 4

3.2 For symmetrically fed antennas of dimensionless cross section, it is well known ([1]) that the currents are distributed sinusoidally from the ends of the antennas. Hence, the current distributions may be written:

$$
\begin{aligned}
I_1 &= I_{1m}\sin k(l_1-s_1) \text{ for } P(\bar{s}_1) \text{ within } (P_{10}P_{12}) \\
&= I_{1m}\sin k(l_1+s_1) \text{ for } P(\bar{s}_1) \text{ within } (P_{11}P_{10}) \\
I_2 &= I_{2m}\sin k(l_2-s_2) \text{ for } P(\bar{s}_2) \text{ within } (P_{20}P_{22}) \\
&= I_{2m}\sin k(l_2+s_2) \text{ for } P(\bar{s}_2) \text{ within } (P_{21}P_{20})
\end{aligned} \tag{3}
$$

Thus, from 2.4(12) and 2.4(13), in terms of the input currents for symmetrically fed antennas, the mutual impedances become:

$$
\begin{aligned}
\bar{\bar{Z}}_{21} = \frac{j30}{k \sin kl_1 \sin kl_2}\Big\{&\oint_{P_{11}}^{P_{10}} \sin k(l_1+s_1)\Big\{\Big[\oint_{P_{21}}^{P_{20}} \sin k(l_2+s_2) + \oint_{P_{20}}^{P_{22}} \sin k(l_2-s_2)\Big] \\
&\bar{\bar{\Delta}}_2\Big[\frac{e^{-jkr_{12}}}{r_{12}}d\bar{s}_1\Big]\cdot d\bar{s}_2\Big\} + \oint_{P_{10}}^{P_{12}} \sin k(l_1-s_1)\Big\{\Big[\oint_{P_{21}}^{P_{20}} \sin k(l_2+s_2) \\
&+\oint_{P_{20}}^{P_{22}} \sin k(l_2-s_2)\Big]\bar{\bar{\Delta}}_2\Big[\frac{e^{-jkr_{12}}}{r_{12}}d\bar{s}_1\Big]\cdot d\bar{s}_2\Big\}\Big\}
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\bar{\bar{Z}}_{12} = \frac{j30}{k \sin kl_1 \sin kl_2}\Big\{&\oint_{P_{21}}^{P_{20}} \sin k(l_2+s_2)\Big\{\Big[\oint_{P_{11}}^{P_{10}} \sin k(l_1+s_1) + \oint_{P_{10}}^{P_{12}} \sin k(l_1-s_1)\Big] \\
&\bar{\bar{\Delta}}_1\Big[\frac{e^{-jkr_{21}}}{r_{21}}d\bar{s}_2\Big]\cdot d\bar{s}_1\Big\} + \oint_{P_{20}}^{P_{22}} \sin k(l_2-s_2)\Big\{\Big[\oint_{P_{11}}^{P_{10}} \sin k(l_1+s_1) + \\
&\oint_{P_{10}}^{P_{12}} \sin k(l_1-s_1)\Big]\bar{\bar{\Delta}}_1\Big[\frac{e^{-jkr_{21}}}{r_{21}}d\bar{s}_2\Big]\cdot d\bar{s}_1\Big\}\Big\}
\end{aligned} \tag{5}
$$

In order that physical symmetry in the final mutual impedances may be assured and in order to facilitate certain integrations, the final form of the impedances again will be taken as

$$
Z_{21} = Z_{12} = \tfrac{1}{2}[\bar{\bar{Z}}_{21}+\bar{\bar{Z}}_{12}] \ . \tag{6}
$$

<u>3.3</u> For reference in carrying out the integrations, the well known integration formulas will be listed:[1]

$$
\int_{z_1}^{z_2} \frac{\cos k(r_0+z)}{r_0}dz = Ci k(r_{02}+z_2) - Ci k(r_{01}+z_1) \tag{1}
$$

$$
\int_{z_1}^{z_2} \frac{\cos k(r_0-z)}{r_0}dz = Ci k(r_{01}-z_1) - Ci k(r_{02}-z_2) \tag{3}
$$

$$\int_{z_1}^{z_2} \frac{\sin k(r_o+z)}{r_o} dz = \mathrm{Si}k(r_{o2}+z_2) - \mathrm{Si}k(r_{o1}+z_1) \qquad (2)$$

$$\int_{z_1}^{z_2} \frac{\sin k(r_o-z)}{r_o} dz = \mathrm{Si}k(r_{o1}-z_1) - \mathrm{Si}k(r_{o2}-z_2) \qquad (4)$$

where

$$r_o = \sqrt{r^2+z^2}, \quad r_{o1} = \sqrt{r^2+z_1^2}, \quad r_{o2} = \sqrt{r^2+z_2^2}$$

and

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos z}{z} dz \ , \ \mathrm{Si}(x) = \int_0^x \frac{\sin z}{z} dz \ .$$

3.4 Letting

$$\bar{g}_1 = \bar{a}_1(\bar{a}_1 \cdot \bar{\Delta}_1 \frac{e^{-jkr_{21}}}{r_{21}}) \text{ and } \bar{g}_2 = \bar{a}_2(\bar{a}_2 \cdot \bar{\Delta}_2 \frac{e^{-jkr_{12}}}{r_{12}}) \ ,$$

the scalar components become

$$g_1 = \bar{a}_1 \cdot \bar{\Delta}_1 \frac{e^{-jkr_{21}}}{r_{21}} = -\bar{a}_1 \cdot \bar{\Delta}_2 \frac{e^{-jkr_{21}}}{r_{21}} = -\bar{\Delta}_2 \cdot \bar{a}_1 \frac{e^{-jkr_{21}}}{r_{21}}$$

and similarly

$$g_2 = -\bar{\Delta}_1 \cdot \bar{a}_2 \frac{e^{-jkr_{12}}}{r_{12}} \ .$$

Hence, from the vector identity for the gradient of the pr oduct of two scalars,

$$\sin k(l_2+s_2)\bar{\Delta}_2(\bar{\Delta}_2 \cdot \frac{e^{-jkr_{12}}}{r_{12}} d\bar{s}_1) = -\sin k(l_2+s_2)\bar{\Delta}_2 g_1 ds_1$$

$$= -\bar{\Delta}_2[g_1 \sin k(l_2+s_2) ds_1] + g_1 \bar{\Delta}_2 \sin k(l_2+s_2) ds_1$$

$$= -\bar{\Delta}_2[g_1 \sin k(l_2+s_2) ds_1] + \bar{a}_2 g_1 k \cos k(l_2+s_2) ds_1 \qquad (1)$$

and similarly,

$$\sin k(l_2-s_2)\bar{\Delta}_2(\bar{\Delta}_2 \cdot \frac{e^{-jkr_{12}}}{r_{12}}) = -\bar{\Delta}_2[g_1 \sin k(l_2-s_2) ds_1] - \bar{a}_2 k g_1 \cos k(l_2-s_2) ds_1 . \qquad (2)$$

Thus,

$$[\oint_{P_{21}}^{P_{20}} \sin k(l_2+s_2) + \oint_{P_{20}}^{P_{22}} \sin k(l_2-s_2)]\bar{\Delta}_2(\bar{\Delta}_2 \cdot \frac{e^{-jkr_{12}}}{r_{12}} d\bar{s}_1) \cdot d\bar{s}_2$$

$$= k[\oint_{P_{21}}^{P_{20}} \cos k(l_2+s_2) - \oint_{P_{20}}^{P_{22}} \cos k(l_2-s_2)]\bar{g}_1 ds_2 \cdot d\bar{s}_1 \ . \qquad (3)$$

Also, integrating by parts:

$$k^2\left[\oint_{P_{21}}^{P_{20}}\sin k(l_2+s_2) + \oint_{P_{20}}^{P_{22}}\sin k(l_2-s_2)\right]\frac{e^{-jkr_{12}}}{r_{12}}\,d\bar{s}_1\cdot d\bar{s}_2$$

$$= k\left\{\bar{a}_2\left[\frac{e^{-jk\acute{r}_{22}}}{\acute{r}_{22}} + \frac{e^{-jk\acute{r}_{21}}}{\acute{r}_{21}} - 2\cos kl_2\,\frac{e^{-jk\acute{r}_{20}}}{\acute{r}_{20}}\right]\right. \tag{4}$$

$$\left. + \left[\oint_{P_{21}}^{P_{20}}\cos k(l_2+s_2) - \oint_{P_{20}}^{P_{22}}\cos k(l_2-s_2)\right]\bar{g}_2\,ds_2\right\}\cdot d\bar{s}_1$$

in which

$\acute{r}_{22}$ is the distance from P_{22} to $P(\bar{s}_1)$

$\acute{r}_{21}$ is the distance from P_{21} to $P(\bar{s}_1)$

$\acute{r}_{20}$ is the distance from P_{20} to $P(\bar{s}_1)$

and in which one lets

$$u = \frac{e^{-jkr_{12}}}{r_{12}} \qquad dv = \sin k(l_2+s_2)ds_2 \qquad dv' = \sin k(l_2-s_2)ds_2$$

$$du = \bar{g}_2\cdot d\bar{s}_2 \qquad v = -\frac{1}{k}\cos k(l_2+s_2) \qquad v' = \frac{1}{k}\cos k(l_2-s_2)\ .$$

Hence

$$\bar{\bar{Z}}_{21} = \frac{j30}{\sin kl_1 \sin kl_2}\left\{\left[\oint_{P_{11}}^{P_{10}}\sin k(l_1+s_1) + \oint_{P_{10}}^{P_{12}}\sin k(l_1-s_1)\right]\bar{a}_2\right.$$

$$\left.\cdot\left[\frac{e^{-jk\acute{r}_{22}}}{\acute{r}_{22}} + \frac{e^{-jk\acute{r}_{21}}}{\acute{r}_{21}} - 2\cos kl_2\frac{e^{-jk\acute{r}_{20}}}{\acute{r}_{20}}\right]d\bar{s}_1\right\}$$

$$+ \frac{j30}{\sin kl_1 \sin kl_2}\left\{\left[\oint_{P_{11}}^{P_{10}}\sin k(l_1+s_1) + \oint_{P_{10}}^{P_{12}}\sin k(l_1-s_1)\right]\bar{a}_1\right. \tag{5}$$

$$\left.\cdot\left[\oint_{P_{21}}^{P_{20}}(\bar{g}_1+\bar{g}_2)\cos k(l_2+s_2)ds_2 - \oint_{P_{20}}^{P_{22}}(\bar{g}_1+\bar{g}_2)\cos k(l_2-s_2)ds_2\right]ds_1\right\}$$

and similarly,

$$\bar{\bar{Z}}_{12} = \frac{j30}{\sin kl_1 \sin kl_2}\left\{\left[\oint_{P_{21}}^{P_{20}}\sin k(l_2+s_2) + \oint_{P_{20}}^{P_{22}}\sin k(l_2-s_2)\right]\bar{a}_1\right.$$

$$\left.\cdot\left[\frac{e^{-jk\acute{r}_{12}}}{\acute{r}_{12}} + \frac{e^{-jk\acute{r}_{11}}}{\acute{r}_{11}} - 2\cos kl_1\frac{e^{-jk\acute{r}_{10}}}{\acute{r}_{10}}\right]d\bar{s}_2\right\}$$

$$+ \frac{j30}{\sin kl_1 \sin kl_2}\left\{\left[\oint_{P_{21}}^{P_{20}}\sin k(l_2+s_2) + \oint_{P_{20}}^{P_{22}}\sin k(l_2-s_2)\right]\bar{a}_2\right. \tag{6}$$

$$\left.\cdot\left[\oint_{P_{11}}^{P_{10}}(\bar{g}_1+\bar{g}_2)\cos k(l_1+s_1)ds_1 - \oint_{P_{10}}^{P_{12}}(\bar{g}_1+\bar{g}_2)\cos k(l_1-s_1)ds_1\right]ds_2\right\}$$

it being understood that the direction cosine must be integrated over both paths.

At this point it should be instructive to examine (6) for a physical interpretation of the various terms involved. A study of the method of deriving 2.3(7) shows that the mutual impedance has been taken in a manner consistent with [1]

$$\bar{\bar{Z}}_{21} = \frac{2\,\Psi_{21}}{I_{2in}I^*_{1in}} \tag{7}$$

in which Ψ_{21} is the mutual complex power given by the integration of the negative electric field from antenna two against the complex conjugate of the current in antenna one.

Schelkunoff[1,17] has shown that the electric field from a sinusoidally distributed current has two components in cylindrical coordinates, namely, an axial component and a radial component:

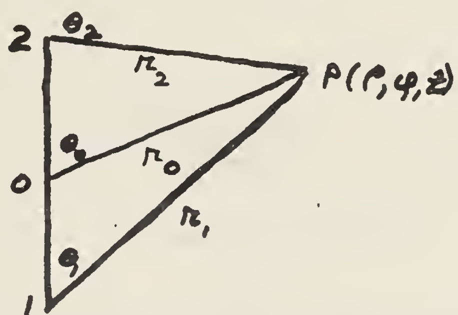


Figure 5

$$\begin{aligned} E_z &= j30I\left[2\cos kl_1\frac{e^{-jkr_0}}{r_0} - \frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_2}}{r_2}\right] \\ E_\rho &= -j30I\left[2\cos kl_1\frac{e^{-jkr_0}}{r_0}\cot\theta_0 - \frac{e^{-jkr_1}}{r_1}\cot\theta_1 - \frac{e^{-jkr_2}}{r_2}\cot\theta_2\right] \end{aligned} \tag{8}$$

Thus, the first term of (6) may be readily identified as arising from the component of the axial field from antenna two taken in the direction of antenna one. It will be shown that the second term is equivalent to the integration of the radial component of the field from two against the current in one, that is, it will be shown that

$$(\bar{g}_1+\bar{g}_2)\cdot d\bar{s}_1 = -[\bar{a}_2 x(\bar{a}_2 x\bar{\Delta}_1 \frac{e^{-jkr_{21}}}{r_{21}})]\cdot d\bar{s}_1 \qquad (9)$$

the vector within the bracket of the right member obviously being normal to $\bar{a}_2$.

Consider:

$$\begin{aligned}(\bar{g}_1+\bar{g}_2)\cdot\bar{a}_1 &= g_1 + (\bar{a}_1\cdot\bar{a}_2)g_2 \\ &= [\bar{a}_1\cdot\bar{\Delta}_1 + (\bar{a}_1\cdot\bar{a}_2)(\bar{a}_2\cdot\bar{\Delta}_2)]\frac{e^{-jkr_{21}}}{r_{21}} \\ &= [\bar{a}_1\cdot\bar{a}_{21}+(\bar{a}_1\cdot\bar{a}_2)(\bar{a}_2\cdot\bar{a}_{12})]\frac{\partial}{\partial r_{21}}\frac{e^{-jkr_{21}}}{r_{21}} \\ &= -[(\bar{a}_1\cdot\bar{a}_2)(\bar{a}_2\cdot\bar{a}_{21}) - \bar{a}_1\cdot\bar{a}_{21}]\frac{\partial}{\partial r_{21}}\frac{e^{-jkr_{21}}}{r_{21}} \\ &= -\bar{a}_1\cdot[\bar{a}_2(\bar{a}_2\cdot\bar{a}_{21}) - (\bar{a}_2\cdot\bar{a}_2)\bar{a}_{21}]\frac{\partial}{\partial r_{21}}\frac{e^{-jkr_{21}}}{r_{21}}\end{aligned}$$

Hence,

$$(\bar{g}_1+\bar{g}_2)\cdot\bar{a}_1 = -\bar{a}_1\cdot[\bar{a}_2 x(\bar{a}_2 x\bar{\Delta}_1)]\frac{e^{-jkr_{21}}}{r_{21}} \qquad (10)$$

and (9) follows.

If $\bar{a}_r$ is defined as

$$\bar{a}_r = \frac{\bar{a}_2 x(\bar{a}_{21} x\bar{a}_2)}{|\bar{a}_{21} x\bar{a}_2|} \qquad (11)$$

in consequence of (8) and (9), (6) may be written:

$$\begin{aligned}\bar{\bar{Z}}_{21} = j\frac{30}{\sin kl_1 \sin kl_2}&\left[\oint_{P_{11}}^{P_{10}} \sin k(l_1+s_1) + \oint_{P_{10}}^{P_{12}} \sin k(l_1-s_1)\right] \\ &\left\{\bar{a}_2\left[\frac{e^{-jkr_{22}}}{r_{22}} + \frac{e^{-jkr_{21}}}{r_{21}} - 2\cos kl_2\frac{e^{-jkr_{20}}}{r_{20}}\right]\right. \qquad (12) \\ - \bar{a}_r\Big[\frac{e^{-jkr_{22}}}{r_{22}}\frac{\bar{r}_{22}\cdot\bar{a}_2}{|\bar{r}_{22} x\bar{a}_2|} &+ \frac{e^{-jkr_{21}}}{r_{21}}\frac{\bar{r}_{21}\cdot\bar{a}_2}{|\bar{r}_{21} x\bar{a}_2|} - 2\cos kl_2\frac{e^{-jkr_{20}}}{r_{20}}\frac{\bar{r}_{20}\cdot\bar{a}_2}{|\bar{r}_{20} x\bar{a}_2|}\Big]\Big\}\cdot d\bar{s}_1\end{aligned}$$

with a similar expression for $\bar{\bar{Z}}_{12}$. However, the radial component of (12) is not readily integrable.

3.6 It should be observed that if $\bar{a}_1=\bar{a}_2$, that is, if the antennas have parallel directions,

$$\bar{a}_r \cdot \bar{a}_1 = \bar{a}_r \cdot \bar{a}_2 = 0 \qquad (1)$$

and the mutual impedances are given by

$$Z_{21} = Z_{12} = \frac{j15}{\sin kl_1 \sin kl_2}(\mathring{Z}_{12} + \mathring{Z}_{21}) \qquad (2)$$

where

$$\mathring{Z}_{21} = [\oint_{P_{11}}^{P_{10}} \sin k(l_1+s_1) + \oint_{P_{10}}^{P_{12}} \sin k(l_1-s_1)] \qquad (3)$$

$$[\frac{e^{-jk\mathring{r}_{22}}}{\mathring{r}_{22}} + \frac{e^{-jk\mathring{r}_{21}}}{\mathring{r}_{21}} - 2\cos kl_2 \frac{e^{-jk\mathring{r}_{20}}}{\mathring{r}_{20}}]ds_1$$

and

$$\mathring{Z}_{12} = [\oint_{P_{21}}^{P_{20}} \sin k(l_2+s_2) + \oint_{P_{20}}^{P_{22}} \sin k(l_2-s_2)] \qquad (4)$$

$$[\frac{e^{-jk\mathring{r}_{12}}}{\mathring{r}_{12}} + \frac{e^{-jk\mathring{r}_{11}}}{\mathring{r}_{11}} - 2\cos kl_1 \frac{e^{-jk\mathring{r}_{10}}}{\mathring{r}_{10}}]ds_2$$

As a matter of fact, after carrying out the integrations for either $\mathring{Z}_{12}$ or $\mathring{Z}_{21}$, the resulting formula is perfectly symmetric in l_1 and l_2. Hence, in this case physical symmetry is assured by carrying out only one set of integrations, that is , one may express

$$Z_{21} = Z_{12} = \frac{j30}{\sin kl_1 \sin kl_2}\mathring{Z}_{21} \qquad (5)$$

Of course, even in the general case, if all integrations could be rigorously completed, in view of 2,4(1), one could be assured of physical symmetry in the resulting formula even though some of the projections which arise may seem to indicate otherwise.

The logical procedure seems to be to integrate rigorously for $\mathring{Z}_{21}$, and in the general case, to write $\mathring{Z}_{12}$ by analogy with $\mathring{Z}_{21}$, and to carry out the integrations of the remaining terms as rigorously and logically possible.

After integrating rigorously for the parallel-staggered case, the integrations will be attempted for the quasi-parallel case, following which, the formulas will be required to reduce to those for the parallel-staggered case and to fulfil such other conditions as become apparent from a study of 3.4(12).

For the quasi-parallel case, let

$$n = \bar{a}_1 \cdot \bar{a}_2 \doteq 1$$

and from the additions of 3.4(5) and 3.4(6), one obtains

$$-j\frac{1}{30}\sin kl_1 \sin kl_2[\bar{Z}_{12}+\bar{Z}_{21}] \doteq n[\dot{Z}_{12} + \dot{Z}_{21}]$$

$$+\int_{P_{21}}^{P_{20}}\int_{P_{11}}^{P_{10}}(g_1+g_2)\sin k(l_1+l_2+s_1+s_2)ds_1 ds_2$$

$$-\int_{P_{21}}^{P_{20}}\int_{P_{10}}^{P_{12}}(g_1+g_2)\sin k(-l_1+l_2+s_1+s_2)ds_1 ds_2 \qquad (6)$$

$$-\int_{P_{20}}^{P_{22}}\int_{P_{11}}^{P_{10}}(g_1+g_2)\sin k(l_1-l_2+s_1+s_2)ds_1 ds_2$$

$$+\int_{P_{20}}^{P_{22}}\int_{P_{10}}^{P_{12}}(g_1+g_2)\sin k(-l_1-l_2+s_1+s_2)ds_1 ds_2$$

<u>3.6</u> To integrate 3.5(3), first write

$$\bar{r}_{22} = \bar{r}_0 - \bar{l}_2 + \bar{s}_1$$

$$\bar{r}_{20} = \bar{r}_0 + \bar{s}_1 \qquad (1)$$

$$\bar{r}_{21} = \bar{r}_0 + \bar{l}_2 + \bar{s}_1$$

and break them into three mutually normal components, respectively.

Thus, if

$$\bar{a}_1 \doteq \bar{a}_z \qquad \bar{a}_1 x \frac{\bar{a}_c x \bar{a}_1}{|\bar{a}_c x \bar{a}_1|} = \bar{a}_y \qquad \frac{\bar{a}_c x \bar{a}_1}{|\bar{a}_c x \bar{a}_1|} = \bar{a}_x \qquad (2)$$

the distances become

$$
\begin{aligned}
\bar{r}_{22} &= \sqrt{[\bar{a}_x\cdot(\bar{r}_c-\bar{l}_2)]^2+[\bar{a}_y\cdot(\bar{r}_c-\bar{l}_2)]^2+[\bar{a}_1\cdot(\bar{r}_c-\bar{l}_2)+s_1]^2} \\
\bar{r}_{20} &= \sqrt{[\bar{a}_x\cdot\bar{r}_c]^2 + [\bar{a}_y\cdot\bar{r}_c]^2 + [\bar{a}_1\cdot\bar{r}_c+s_1]^2} \qquad (3) \\
\bar{r}_{21} &= \sqrt{[\bar{a}_x\cdot(\bar{r}_c+\bar{l}_2)]^2+[\bar{a}_y\cdot(\bar{r}_c+\bar{l}_2)]^2+[\bar{a}_1\cdot(\bar{r}_c+\bar{l}_2)+s_1]^2}
\end{aligned}
$$

To reduce the lengths to the form for applying formulas 3.3(1) to 3.3(4), choose the origin, respectively, such that

$$
\begin{aligned}
&\text{for } \bar{r}_{22}, \quad z = s_1+\bar{a}_1\cdot(\bar{r}_c-\bar{l}_2) \\
&\text{for } \bar{r}_{20}, \quad z = s_1+\bar{a}_1\cdot\bar{r}_c \qquad (4) \\
&\text{for } \bar{r}_{21}, \quad z = s_1+\bar{a}_1\cdot(\bar{r}_c+\bar{l}_2) .
\end{aligned}
$$

Equation 3.6(5) then becomes

$$
\begin{aligned}
\bar{Z}_{21} = & \Bigg\{\int_{-l_1+\bar{a}_1\cdot(\bar{r}_c-\bar{l}_2)}^{\bar{a}_1\cdot(\bar{r}_c-\bar{l}_2)} \sin k[l_1-\bar{a}_1\cdot(\bar{r}_c-\bar{l}_2)] \\
& + \int_{\bar{a}_1\cdot(\bar{r}_c-\bar{l}_2)}^{l_1+\bar{a}_1\cdot(\bar{r}_c-\bar{l}_2)} \sin k[l_1+\bar{a}_1\cdot(\bar{r}_c-\bar{l}_2) - z]\Bigg\}\frac{e^{-jk\bar{r}_{22}}}{\bar{r}_{22}}dz \\
& + \Bigg\{\int_{-l_1+\bar{a}_1\cdot(\bar{r}_c+\bar{l}_2)}^{\bar{a}_1\cdot(\bar{r}_c+\bar{l}_2)} \sin k[l_1-\bar{a}_1\cdot(\bar{r}_c+\bar{l}_2)+z] \qquad (5) \\
& + \int_{\bar{a}_1\cdot(\bar{r}_c+\bar{l}_2)}^{l_1+\bar{a}_1\cdot(\bar{r}_c+\bar{l}_2)} \sin k[l_1+\bar{a}_1\cdot(\bar{r}_c+\bar{l}_2)-z]\Bigg\}\frac{e^{-jk\bar{r}_{21}}}{\bar{r}_{21}} dz \\
& -2\cos kl_2\Bigg\{\int_{-\bar{l}_1+\bar{a}_1\cdot\bar{r}_c}^{\bar{a}_1\cdot\bar{r}_c} \sin k[l_1-\bar{a}_1\cdot\bar{r}_c+z] \\
& + \int_{\bar{a}_1\cdot\bar{r}_c}^{l_1+\bar{a}_1\cdot\bar{r}_c} \sin k[l_1+\bar{a}_1\cdot\bar{r}_c-z]\Bigg\}\frac{e^{-jk\bar{r}_{20}}}{\bar{r}_{20}} dz
\end{aligned}
$$

3.7 Since all terms of 3.6(5) are of the type

$$
B = \int_{z_1}^{z_2} \sin k(A+z)\frac{e^{-jkr_0}}{r_0}dz \qquad (1)
$$

time will be saved by integrating (1) by 3.3(1) to 3.3(4) and using the result as an integration formula. Thus

$$B = \frac{1}{2}\Big\{\sin A[\mathrm{Cik}(r_{02}+z_2)-\mathrm{Cik}(r_{01}+z_1)+\mathrm{Cik}(r_{01}-z_1)-\mathrm{Cik}(r_{02}-z_2)]$$
$$+\cos A[\mathrm{Sik}(r_{02}+z_2)-\mathrm{Sik}(r_{01}+z_1)-\mathrm{Sik}(r_{01}-z_1)+\mathrm{Sik}(r_{02}-z_2)]\Big\} \tag{2}$$
$$+j\frac{1}{2}\Big\{\cos A[\mathrm{Cik}(r_{02}+z_2)-\mathrm{Cik}(r_{01}+z_1)-\mathrm{Cik}(r_{01}-z_1)+\mathrm{Cik}(r_{02}-z_2)]$$
$$-\sin A[\mathrm{Sik}(r_{02}+z_2)-\mathrm{Sik}(r_{01}+z_1)+\mathrm{Sik}(r_{01}-z_1)-\mathrm{Sik}(r_{02}-z_2)]\Big\}$$

For

$$B = \int_{z_1}^{z_2} \sin k(A-z)\frac{e^{-jkr_0}}{r_0}dz \tag{3}$$

use (2) with cosA replaced by -cosA.

For

$$B = \int_{z_1}^{z_2} \cos k(A+z)\frac{e^{-jkr_0}}{r_0}dz \tag{4}$$

use (2) with sinA replaced by cos(-A) and cosA by sin(-A) .

For

$$B = \int_{z_1}^{z_2} \cos k(A-z)\frac{e^{-jkr_0}}{r_0}dz \tag{5}$$

replace sinA by cosA and replace cosA by sinA.

3.2 For additional aid in the writing of the formula for $\dot{Z}_{21}$,

let

$$\dot{r}_{2211} = \sqrt{[\bar{a}_x\cdot(\bar{r}_c+\bar{l}_2)]^2+[\bar{a}_y\cdot(\bar{r}_c+\bar{l}_2)]^2+[\bar{a}_1\cdot(\bar{r}_c+\bar{l}_2-\bar{l}_1)]^2} \tag{1}$$

$$\dot{r}_{2210} = \sqrt{[\bar{a}_x\cdot(\bar{r}_c-\bar{l}_2)]^2+[\bar{a}_y\cdot(\bar{r}_c-\bar{l}_2)]^2+[\bar{a}_1\cdot(\bar{r}_c-\bar{l}_2)]^2}$$

$$\dot{r}_{2011} = \sqrt{[\bar{a}_x\cdot\bar{r}_c]^2+[\bar{a}_y\cdot\bar{r}_c]^2+[\bar{a}_1\cdot(\bar{r}_c-\bar{l}_1)]^2} \tag{3}$$

$$\dot{r}_{2212} = \sqrt{[\bar{a}_x\cdot(\bar{r}_c-\bar{l}_2)]^2+[\bar{a}_y\cdot(\bar{r}_c-\bar{l}_2)]^2+[\bar{a}_1\cdot(\bar{r}_c-\bar{l}_2+\bar{l}_1)]^2} \tag{4}$$

$$\dot{r}_{2010} = \sqrt{[\bar{a}_x\cdot\bar{r}_c]^2+[\bar{a}_y\cdot\bar{r}_c]^2+[\bar{a}_1\cdot\bar{r}_c]^2} \tag{5}$$

$$\dot{r}_{2111} = \sqrt{[\bar{a}_x\cdot(\bar{r}_c+\bar{l}_2)]^2+[\bar{a}_y\cdot(\bar{r}_c+\bar{l}_2)]^2+[\bar{a}_1\cdot(\bar{r}_c+\bar{l}_2-\bar{l}_1)]^2} \tag{6}$$

$$\dot{r}_{2012} = \sqrt{[\bar{a}_x\cdot\bar{r}_c]^2+[\bar{a}_y\cdot\bar{r}_c]^2+[\bar{a}_1\cdot(\bar{r}_c+\bar{l}_1)]^2} \tag{7}$$

$$\hat{r}_{2210} = \sqrt{[\bar{a}_x\cdot(\bar{r}_o+\bar{l}_2)]^2+[\bar{a}_y\cdot(\bar{r}_o+\bar{l}_2)]^2+[\bar{a}_1\cdot(\bar{r}_o+\bar{l}_2)]^2} \qquad (8)$$

$$\hat{r}_{2211} = \sqrt{[\bar{a}_x\cdot(\bar{r}_o+\bar{l}_2)]^2+[\bar{a}_y\cdot(\bar{r}_o+\bar{l}_2)]^2+[\bar{a}_1\cdot(\bar{r}_o+\bar{l}_2+\bar{l}_1)]^2} \qquad (9)$$

that is,

$\hat{r}_{2211}$ is the diagonal length from P_{22} to P_{11},

$\hat{r}_{2210}$ is the diagonal length from P_{22} to P_{10}, etc.

Furthermore, let

$$h_1 = \bar{a}_1\cdot\bar{r}_o$$

and let

$$\begin{aligned}
z_{21} &= \bar{a}_1\cdot(\bar{r}_o-\bar{l}_1-\bar{l}_2) &&= h_1-l_1-nl_2 \\
z_{22} &= \bar{a}_1\cdot(\bar{r}_o-\bar{l}_2) &&= h_1-nl_2 \\
z_{23} &= \bar{a}_1\cdot(\bar{r}_o-\bar{l}_1) &&= h_1-l_1 \\
z_{24} &= \bar{a}_1\cdot(\bar{r}_o+\bar{l}_1-\bar{l}_2) &&= h_1+l_1-nl_2 \\
z_{25} &= \bar{a}_1\cdot(\bar{r}_o) &&= h_1 \\
z_{26} &= \bar{a}_1\cdot(\bar{r}_o-\bar{l}_1+\bar{l}_2) &&= h_1-l_1+nl_2 \\
z_{27} &= \bar{a}_1\cdot(\bar{r}_o+\bar{l}_1) &&= h_1+l_1 \\
z_{28} &= \bar{a}_1\cdot(\bar{r}_o+\bar{l}_2) &&= h_1+nl_2 \\
z_{29} &= \bar{a}_1\cdot(\bar{r}_o+\bar{l}_1+\bar{l}_2) &&= h_1+l_1+nl_2
\end{aligned} \qquad (10)$$

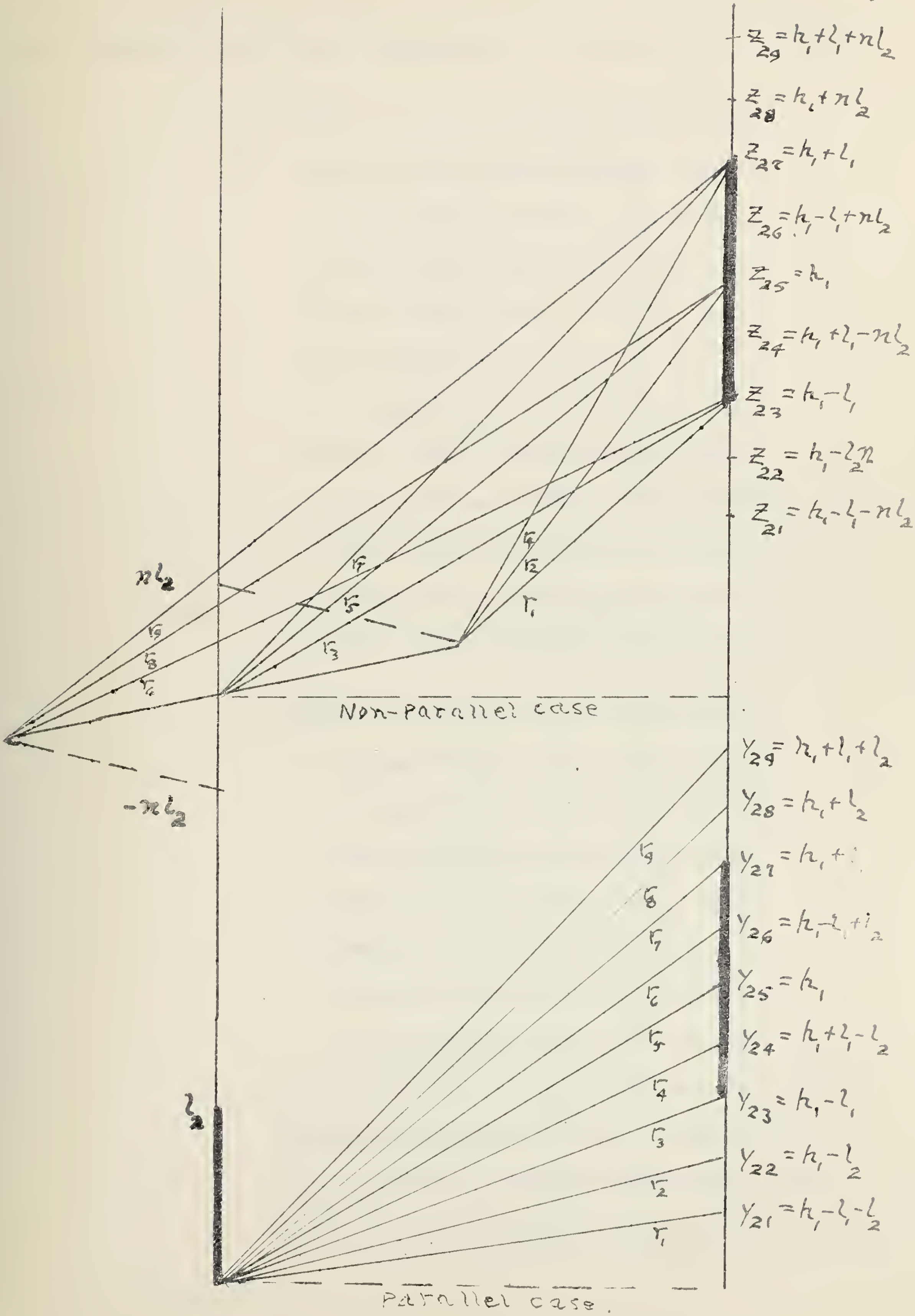
Also, let

$$y_{2i} = z_{2i} \text{ for } i = 1, \ldots 9 \qquad (11)$$

and let

$$\begin{aligned}
\bar{L}_{i_1}^2 &= \hat{r}_i - z_{2i} \\
\bar{L}_{i2}^2 &= \hat{r}_i + z_{2i}
\end{aligned} \qquad (12)$$

in which $\hat{r}_i$ is the i^{th} diagonal for $i = 1, \ldots 9$. In case $n=1$, replace $\bar{L}_i^2$ by L_i^2 and replace $\hat{r}_i$ by r_i. The following sketch will aid in choosing the proper symbols for the formula:



$Z_{29} = h_1 + l_1 + nl_2$
$Z_{28} = h_1 + nl_2$
$Z_{27} = h_1 + l_1$
$Z_{26} = h_1 - l_1 + nl_2$
$Z_{25} = h_1$
$Z_{24} = h_1 + l_1 - nl_2$
$Z_{23} = h_1 - l_1$
$Z_{22} = h_1 - l_2 n$
$Z_{21} = h_1 - l_1 - nl_2$
nl_2
$-nl_2$
Non-parallel case
$Y_{29} = h_1 + l_1 + l_2$
$Y_{28} = h_1 + l_2$
$Y_{27} = h_1 + l_1$
$Y_{26} = h_1 - l_1 + l_2$
$Y_{25} = h_1$
$Y_{24} = h_1 + l_1 - l_2$
$Y_{23} = h_1 - l_1$
$Y_{22} = h_1 - l_2$
$Y_{21} = h_1 - l_1 - l_2$
l_2
Parallel case.

3.9 Applying 3.7(2) to 3.6(6) and using the notation of 3.8, after slight algebraic and trigonometric manipulation, one obtains:

$$
\begin{aligned}
\hat{Z}_{21} = \frac{1}{2}\Big\{ & \sin kz_{24}\left[\mathrm{Cik}\,\bar{\bar{L}}^2_{21}-\mathrm{Cik}\,\bar{\bar{L}}^2_{41}+\mathrm{Cik}\,\bar{\bar{L}}^2_{42}-\mathrm{Cik}\,\bar{\bar{L}}^2_{22}\right] \\
& +\cos kz_{24}\left[\mathrm{Sik}\,\bar{\bar{L}}^2_{21}-\mathrm{Sik}\,\bar{\bar{L}}^2_{41}-\mathrm{Sik}\,\bar{\bar{L}}^2_{42}+\mathrm{Sik}\,\bar{\bar{L}}^2_{22}\right] \\
& +\sin kz_{29}\left[\mathrm{Cik}\,\bar{\bar{L}}^2_{81}-\mathrm{Cik}\,\bar{\bar{L}}^2_{91}+\mathrm{Cik}\,\bar{\bar{L}}^2_{92}-\mathrm{Cik}\,\bar{\bar{L}}^2_{82}\right] \\
& +\cos kz_{29}\left[\mathrm{Sik}\,\bar{\bar{L}}^2_{81}-\mathrm{Sik}\,\bar{\bar{L}}^2_{91}-\mathrm{Sik}\,\bar{\bar{L}}^2_{92}+\mathrm{Sik}\,\bar{\bar{L}}^2_{82}\right] \\
& -\sin kz_{21}\left[\mathrm{Cik}\,\bar{\bar{L}}^2_{11}-\mathrm{Cik}\,\bar{\bar{L}}^2_{21}+\mathrm{Cik}\,\bar{\bar{L}}^2_{22}-\mathrm{Cik}\,\bar{\bar{L}}^2_{12}\right] \\
& -\cos kz_{21}\left[\mathrm{Sik}\,\bar{\bar{L}}^2_{11}-\mathrm{Sik}\,\bar{\bar{L}}^2_{21}-\mathrm{Sik}\,\bar{\bar{L}}^2_{22}+\mathrm{Sik}\,\bar{\bar{L}}^2_{12}\right] \\
& -\sin kz_{26}\left[\mathrm{Cik}\,\bar{\bar{L}}^2_{61}-\mathrm{Cik}\,\bar{\bar{L}}^2_{81}+\mathrm{Cik}\,\bar{\bar{L}}^2_{82}-\mathrm{Cik}\,\bar{\bar{L}}^2_{62}\right] \\
& -\cos kz_{26}\left[\mathrm{Sik}\,\bar{\bar{L}}^2_{61}-\mathrm{Sik}\,\bar{\bar{L}}^2_{81}-\mathrm{Sik}\,\bar{\bar{L}}^2_{82}+\mathrm{Sik}\,\bar{\bar{L}}^2_{62}\right] \\
& -(\sin ky_{29}+\sin ky_{24})\left[\mathrm{Cik}\,\bar{\bar{L}}^2_{51}-\mathrm{Cik}\,\bar{\bar{L}}^2_{71}+\mathrm{Cik}\,\bar{\bar{L}}^2_{72}-\mathrm{Cik}\,\bar{\bar{L}}^2_{52}\right] \\
& -(\cos ky_{29}+\cos ky_{24})\left[\mathrm{Sik}\,\bar{\bar{L}}^2_{51}-\mathrm{Sik}\,\bar{\bar{L}}^2_{71}-\mathrm{Sik}\,\bar{\bar{L}}^2_{72}+\mathrm{Sik}\,\bar{\bar{L}}^2_{52}\right] \\
& +(\sin ky_{26}+\sin ky_{21})\left[\mathrm{Cik}\,\bar{\bar{L}}^2_{31}-\mathrm{Cik}\,\bar{\bar{L}}^2_{51}+\mathrm{Cik}\,\bar{\bar{L}}^2_{52}-\mathrm{Cik}\,\bar{\bar{L}}^2_{32}\right] \\
& +(\cos ky_{26}+\cos ky_{21})\left[\mathrm{Sik}\,\bar{\bar{L}}^2_{31}-\mathrm{Sik}\,\bar{\bar{L}}^2_{51}-\mathrm{Sik}\,\bar{\bar{L}}^2_{52}+\mathrm{Sik}\,\bar{\bar{L}}^2_{32}\right]\Big\} \\
-j\frac{1}{2}\Big\{ & \sin kz_{24}\left[\mathrm{Sik}\,\bar{\bar{L}}^2_{21}-\mathrm{Sik}\,\bar{\bar{L}}^2_{41}+\mathrm{Sik}\,\bar{\bar{L}}^2_{42}-\mathrm{Sik}\,\bar{\bar{L}}^2_{22}\right] \\
& -\cos kz_{24}\left[\mathrm{Cik}\,\bar{\bar{L}}^2_{21}-\mathrm{Cik}\,\bar{\bar{L}}^2_{41}-\mathrm{Cik}\,\bar{\bar{L}}^2_{42}+\mathrm{Cik}\,\bar{\bar{L}}^2_{22}\right] \\
& +\sin kz_{29}\left[\mathrm{Sik}\,\bar{\bar{L}}^2_{81}-\mathrm{Sik}\,\bar{\bar{L}}^2_{91}+\mathrm{Sik}\,\bar{\bar{L}}^2_{92}-\mathrm{Sik}\,\bar{\bar{L}}^2_{82}\right] \\
& -\cos kz_{29}\left[\mathrm{Cik}\,\bar{\bar{L}}^2_{81}-\mathrm{Cik}\,\bar{\bar{L}}^2_{91}-\mathrm{Cik}\,\bar{\bar{L}}^2_{92}+\mathrm{Cik}\,\bar{\bar{L}}^2_{82}\right] \\
& -\sin kz_{21}\left[\mathrm{Sik}\,\bar{\bar{L}}^2_{11}-\mathrm{Sik}\,\bar{\bar{L}}^2_{21}+\mathrm{Sik}\,\bar{\bar{L}}^2_{22}-\mathrm{Sik}\,\bar{\bar{L}}^2_{12}\right] \\
& +\cos kz_{21}\left[\mathrm{Cik}\,\bar{\bar{L}}^2_{11}-\mathrm{Cik}\,\bar{\bar{L}}^2_{21}-\mathrm{Cik}\,\bar{\bar{L}}^2_{22}+\mathrm{Cik}\,\bar{\bar{L}}^2_{12}\right] \\
& -\sin kz_{26}\left[\mathrm{Sik}\,\bar{\bar{L}}^2_{61}-\mathrm{Sik}\,\bar{\bar{L}}^2_{81}+\mathrm{Sik}\,\bar{\bar{L}}^2_{82}-\mathrm{Sik}\,\bar{\bar{L}}^2_{62}\right] \\
& +\cos kz_{26}\left[\mathrm{Cik}\,\bar{\bar{L}}^2_{61}-\mathrm{Cik}\,\bar{\bar{L}}^2_{81}-\mathrm{Cik}\,\bar{\bar{L}}^2_{82}+\mathrm{Cik}\,\bar{\bar{L}}^2_{62}\right] \\
& -(\sin ky_{29}+\sin ky_{24})\left[\mathrm{Sik}\,\bar{\bar{L}}^2_{51}-\mathrm{Sik}\,\bar{\bar{L}}^2_{71}+\mathrm{Sik}\,\bar{\bar{L}}^2_{72}-\mathrm{Sik}\,\bar{\bar{L}}^2_{52}\right] \\
& +(\cos ky_{29}+\cos ky_{24})\left[\mathrm{Cik}\,\bar{\bar{L}}^2_{51}-\mathrm{Cik}\,\bar{\bar{L}}^2_{71}-\mathrm{Cik}\,\bar{\bar{L}}^2_{72}+\mathrm{Cik}\,\bar{\bar{L}}^2_{52}\right] \\
& +(\sin ky_{26}+\sin ky_{21})\left[\mathrm{Sik}\,\bar{\bar{L}}^2_{31}-\mathrm{Sik}\,\bar{\bar{L}}^2_{51}+\mathrm{Sik}\,\bar{\bar{L}}^2_{52}-\mathrm{Sik}\,\bar{\bar{L}}^2_{32}\right] \\
& -(\cos ky_{26}+\cos ky_{21})\left[\mathrm{Cik}\,\bar{\bar{L}}^2_{31}-\mathrm{Cik}\,\bar{\bar{L}}^2_{51}-\mathrm{Cik}\,\bar{\bar{L}}^2_{52}+\mathrm{Cik}\,\bar{\bar{L}}^2_{32}\right]\Big\}
\end{aligned}
$$

It is again emphasized that for parallel-staggered antennas,

$$Z_{21} = Z_{12} = \frac{j30}{\sin kl_1 \sin kl_2}\mathring{Z}_{21} \; . \qquad (2)$$

3.10 From 3.1(2),

$$\begin{aligned} \mathring{\bar{r}}_{12} &= -\bar{r}_c-\bar{l}_1+\bar{s}_2 \\ \mathring{\bar{r}}_{10} &= -\bar{r}_c+\bar{s}_2 \\ \mathring{\bar{r}}_{11} &= -\bar{r}c+\bar{l}_1+\bar{s}_2 \end{aligned} \qquad (1)$$

Letting

$$\bar{a}_2=\bar{a}_z \; , \; \bar{a}_2 x \frac{\bar{a}_c x \bar{a}_2}{|\bar{a}_c x \bar{a}_2|} = \bar{a}_y \; , \; \frac{\bar{a}_c x \bar{a}_y}{|\bar{a}_c x \bar{a}_y|} = \bar{a}_x \qquad (2)$$

the distances become

$$\begin{aligned} \mathring{r}_{12} &= \sqrt{[\bar{a}_x\cdot(-\bar{r}_c-\bar{l}_1)]^2+[\bar{a}_y\cdot(-\bar{r}_c-\bar{l}_1)]^2+[\bar{a}_2\cdot(-\bar{r}_c-\bar{l}_1)+s_2]^2} \\ \mathring{r}_{10} &= \sqrt{[\bar{a}_x\cdot(-\bar{r}_c)]^2+[\bar{a}_y\cdot(-\bar{r}_c)]^2+[\bar{a}_2\cdot(-\bar{r}_c)+s_2]^2} \\ \mathring{r}_{11} &= \sqrt{[\bar{a}_x\cdot(-\bar{r}_c+\bar{l}_1)]^2+[\bar{a}_y\cdot(-\bar{r}_c+\bar{l}_1)]^2+[\bar{a}_2\cdot(-\bar{r}_c+\bar{l}_1)+s_2]^2} \end{aligned} \qquad (3)$$

Hence, if

$$h_2 = -\bar{a}_2\cdot\bar{r}_c \; .$$

equation 3.4(9) may be integrated for $\mathring{Z}_{12}$ by analogy with $\mathring{Z}_{21}$, provided h_1 is replaced with h_2 and l_1 is interchanged with l_2 in 3.8(10) to (12). In other words, write

$$\begin{aligned} z_{11} &= h_2-l_2-nl_1 \\ z_{12} &= h_2-nl_1 \\ z_{13} &= h_2-l_2 \\ z_{14} &= h_2+l_2-nl_1 \\ z_{15} &= h_2 \\ z_{16} &= h_2-l_2+nl_1 \\ z_{17} &= h_2+l_2 \\ z_{18} &= h_2+nl_1 \end{aligned} \qquad (4)$$

with

$$y_{1i} = z_{1i} \text{ for } i=1, \ldots 9$$

and with $\bar{\bar{L}}^1_i$ and L^1_i defined as in 3.8(12).

Thus, to write $\acute{Z}_{12}$, in 3.9(1) make the following replacements:

$$z_{2i} \text{ by } z_{1i}$$

$$y_{2i} \text{ by } y_{1i}$$

$$\bar{\bar{L}}^2_{ij} \text{ by } \bar{\bar{L}}^1_{ij} \, .$$

Examination of the resulting formula for $\acute{Z}_{12}$ verifies 3.9([illegible]).

[illegible] Having integrated the first two terms of 3.8(8), an attempt must be made to integrate the remaining terms. To proceed, for those terms involving

$$\bar{a}_2 \cdot \bar{\Delta}_2 \frac{e^{-jkr_{12}}}{r_{12}}$$

integrate first with respect to s_2, and for those terms involving

$$\bar{a}_1 \cdot \bar{\Delta}_1 \frac{e^{-jkr_{21}}}{r_{21}}$$

integrate first with respect to s_1. In each case, integrate by parts letting

$$dv = \bar{\Delta}_2 \frac{e^{-jkr_{12}}}{r_{12}} \cdot d\bar{s}_2$$

and

$$dv = \bar{\Delta}_1 \frac{e^{-jkr_{21}}}{r_{21}} \cdot d\bar{s}_1$$

respectively. Thus, after slight trigonometric manipulation,

$$-\frac{j}{30}\sin kl_1 \sin kl_2(\overset{=}{Z}_{21}+\overset{=}{Z}_{12}) = n(\overset{\circ}{Z}_{21} + \overset{\circ}{Z}_{12})$$

$$+\oint_{P_{11}}^{P_{10}}[2\cos kl_2\frac{e^{-jk\overset{\circ}{r}_{20}}}{\overset{\circ}{r}_{20}} -(\frac{e^{-jk\overset{\circ}{r}_{22}}}{\overset{\circ}{r}_{22}} + \frac{e^{-jk\overset{\circ}{r}_{21}}}{\overset{\circ}{r}_{21}})]\sin k(l_1+s_1)ds_1$$

$$+\oint_{P_{10}}^{P_{12}}[\frac{e^{-jk\overset{\circ}{r}_{21}}}{\overset{\circ}{r}_{21}} + \frac{e^{-jk\overset{\circ}{r}_{22}}}{\overset{\circ}{r}_{22}} - 2\cos kl_2\frac{e^{-jk\overset{\circ}{r}_{20}}}{\overset{\circ}{r}_{20}}]\sin k(-l_1+s_1)ds_1$$

$$+\oint_{P_{21}}^{P_{20}}[2\cos kl_1\frac{e^{-jk\overset{\circ}{r}_{10}}}{\overset{\circ}{r}_{10}} - \frac{e^{-jk\overset{\circ}{r}_{11}}}{\overset{\circ}{r}_{11}} - \frac{e^{-jk\overset{\circ}{r}_{12}}}{\overset{\circ}{r}_{12}}]\sin k(l_2+s_2)ds_2$$

$$+\oint_{P_{20}}^{P_{22}}[\frac{e^{-jk\overset{\circ}{r}_{12}}}{\overset{\circ}{r}_{12}} + \frac{e^{-jk\overset{\circ}{r}_{11}}}{\overset{\circ}{r}_{11}} - 2\cos kl_1\frac{e^{-jk\overset{\circ}{r}_{10}}}{\overset{\circ}{r}_{10}}]\sin k(-l_2+s_2)ds_2$$

$$-2k\oint_{P_{21}}^{P_{20}}\oint_{P_{11}}^{P_{10}}\frac{e^{-jkr_{12}}}{r_{12}}\cos k(l_1+l_2+s_1+s_2)ds_1ds_2$$

$$+2k\oint_{P_{21}}^{P_{20}}\oint_{P_{10}}^{P_{12}}\frac{e^{-jkr_{12}}}{r_{12}}\cos k(l_2-l_1+s_1+s_2)ds_1ds_2$$

$$+2k\oint_{P_{20}}^{P_{22}}\oint_{P_{11}}^{P_{10}}\frac{e^{-jkr_{12}}}{r_{12}}\cos k(l_1-l_2+s_1+s_2)ds_1ds_2$$

$$-2k\oint_{P_{20}}^{P_{22}}\oint_{P_{10}}^{P_{12}}\frac{e^{-jkr_{12}}}{r_{12}}\cos k(-l_1-l_2+s_1+s_2)ds_1ds_2$$

Therefore

$$-\frac{j}{30}\sin kl_1 \sin kl_2(\overset{=}{Z}_{21}+\overset{=}{Z}_{12}) = (n-1)(\overset{\circ}{Z}_{21}+\overset{\circ}{Z}_{12})$$

$$-2k\oint_{P_{21}}^{P_{20}}\oint_{P_{11}}^{P_{10}}\frac{e^{-jkr_{12}}}{r_{12}}\cos k(l_1+l_2+s_1+s_2)ds_1ds_2$$

$$+2k\oint_{P_{21}}^{P_{20}}\oint_{P_{10}}^{P_{12}}\frac{e^{-jkr_{12}}}{r_{12}}\cos k(l_2-l_1+s_1+s_2)ds_1ds_2$$

$$+2k\oint_{P_{20}}^{P_{22}}\oint_{P_{11}}^{P_{10}}\frac{e^{-kjr_{12}}}{r_{12}}\cos k(l_1-l_2+s_1+s_2)ds_1ds_2$$

$$-2k\oint_{P_{20}}^{P_{22}}\oint_{P_{10}}^{P_{12}}\frac{e^{-jkr_{12}}}{r_{12}}\cos k(-l_1-l_2+s_1+s_2)ds_1ds_2$$

The first integration of the four double line integrals can be carried out by 3.7(4). However, the second integration will be impossible to carry out analytically by a rigorous method. Hence, as previously mentioned, it is preferable to employ both possible orders of integration. To this end, let

$$\frac{1}{k(n+1)}(\hat{Z}_{12}+\hat{Z}'_{12}) = \frac{1}{k(n+1)}(\hat{Z}_{21}+\hat{Z}''_{21}) =$$

$$-\oint_{P_{21}}^{P_{20}}\oint_{P_{11}}^{P_{10}}\frac{e^{-jkr_{12}}}{r_{12}}\cos k(l_1+l_2+s_1+s_2)ds_1ds_2$$

$$+\oint_{P_{21}}^{P_{20}}\oint_{P_{10}}^{P_{12}}\frac{e^{-jkr_{12}}}{r_{12}}\cos k(l_2-l_1+s_1+s_2)ds_1ds_2 \tag{3}$$

$$+\oint_{P_{20}}^{P_{22}}\oint_{P_{11}}^{P_{10}}\frac{e^{-jkr_{12}}}{r_{12}}\cos k(l_1-l_2+s_1+s_2)ds_1ds_2$$

$$-\oint_{P_{20}}^{P_{22}}\oint_{P_{10}}^{P_{12}}\frac{e^{-jkr_{12}}}{r_{12}}\cos k(-l_1-l_2+s_1+s_2)ds_1ds_2 \; .$$

Obviously, for $\bar{a}_1\cdot\bar{a}_2 = 1$,

$$\frac{\sin kl_1 \sin kl_2}{j30} Z_{21} = \frac{1}{2}(\hat{Z}_{21} + \hat{Z}''_{21}) \; . \tag{4}$$

Equation (4) is a pertinent restriction which will be imposed upon the final integration of (3).

Proceeding with the integration of (3) first in s_1, recall that

$$\bar{r}_{21} = \bar{r}_0+\bar{s}_1-\bar{s}_2 \; .$$

Breaking $\bar{r}_{21}$ into three mutually normal components as in 3.6(2),

$$r_{21} = \sqrt{[\bar{a}_x\cdot(\bar{r}_c-\bar{s}_2)]^2+[\bar{a}_y\cdot(\bar{r}_0-\bar{s}_2)]^2+[\bar{a}_1\cdot(\bar{r}_0-\bar{s}_2)+s_1]^2} \tag{5}$$

Hence, let

$$z = h_1-ns_2+s_1 \tag{6}$$

and (3) becomes

$$\frac{1}{n}(\hat{Z}_{21}+\hat{Z}''_{21}) = -k\oint_{P_{21}}^{P_{20}}\oint_{h_1-ns_2-l_1}^{h_1-ns_2}\frac{e^{-jkr_{21}}}{r_{21}}\cos k[l_1+l_2-h_1+(n+1)s_2+z]dzds_2$$

$$+k\oint_{P_{21}}^{P_{20}}\oint_{h_1-ns_2}^{h_1-ns_2+l_1}\frac{e^{-jkr_{21}}}{r_{21}}\cos k[l_2-l_1-h_1+(n+1)s_2+z]dzds_2 \tag{7}$$

$$+k\oint_{P_{20}}^{P_{22}}\oint_{h_1-ns_2-l_1}^{h_1-ns_2}\frac{e^{-jkr_{21}}}{r_{21}}\cos k[l_1-l_2-h_1+(n+1)s_2+z]dzds_2$$

$$-k\oint_{P_{20}}^{P_{22}}\oint_{h_1-ns_2}^{h_1-ns_2+l_1}\frac{e^{-jkr_{21}}}{r_{21}}\cos k[-l_1-l_2-h_1+(n+1)s_2+z]dzds_2$$

Applying formula 3.7(4) to equation (7):

$$\frac{2}{k(n+1)}(z'_{21}+z''_{21}) =$$

$$-\int_{-l_2}^{0}\cos k[l_1+l_2-h_1+(n+1)s_2][\mathrm{Cik}(r'_{10}+h_1-ns_2)-\mathrm{Cik}(r'_{11}+h_1-l_1-ns_2) \qquad [1]$$
$$+\mathrm{Cik}(r'_{11}-h_1+l_1+ns_2)-\mathrm{Cik}(r'_{10}-h_1+ns_2)]ds_2$$

$$+\int_{-l_2}^{0}\sin k[l_1+l_2-h_1+(n+1)s_2][\mathrm{Sik}(r'_{10}+h_1-ns_2)-\mathrm{Sik}(r'_{11}+h_1-l_1-ns_2) \qquad [2]$$
$$-\mathrm{Sik}(r'_{11}-h_1+l_1+ns_2)+\mathrm{Sik}(r'_{10}-h_1+ns_2)]ds_2$$

$$+\int_{-l_2}^{0}\cos k[l_1-l_2+h_1-(n+1)s_2][\mathrm{Cik}(r'_{10}-h_1+ns_2)-\mathrm{Cik}(r'_{12}-h_1-l_1+ns_2) \qquad [3]$$
$$+\mathrm{Cik}(r'_{12}+h_1+l_1-ns_2)-\mathrm{Cik}(r'_{10}+h_1-ns_2)]ds_2$$

$$-\int_{-l_2}^{0}\sin k[l_1-l_2+h_1-(n+1)s_2][\mathrm{Sik}(r'_{10}-h_1+ns_2)-\mathrm{Sik}(r'_{12}-h_1-l_1+ns_2) \qquad [4]$$
$$-\mathrm{Sik}(r'_{12}+h_1+l_1-ns_2)+\mathrm{Sik}(r'_{10}+h_1-ns_2)]ds_2 \qquad (2a)$$

$$+\int_{0}^{l_2}\cos k[l_1-l_2-h_1-(n+1)s_2][\mathrm{Cik}(r'_{10}+h_1-ns_2)-\mathrm{Cik}(r'_{11}+h_1-l_1-ns_2) \qquad [5]$$
$$+\mathrm{Cik}(r'_{11}-h_1+l_1+ns_2)-\mathrm{Cik}(r'_{10}-h_1+ns_2)]ds_2$$

$$-\int_{0}^{l_2}\sin k[l_1-l_2-h_1+(n+1)s_2][\mathrm{Sik}(r'_{10}+h_1-ns_2)-\mathrm{Sik}(r'_{11}+h_1-l_1-ns_2) \qquad [6]$$
$$-\mathrm{Sik}(r'_{11}-h_1+l_1+ns_2)+\mathrm{Sik}(r'_{10}-h_1+ns_2)]ds_2$$

$$-\int_{0}^{l_2}\cos k[l_1+l_2+h_1-(n+1)s_2][\mathrm{Cik}(r'_{10}-h_1+ns_2)-\mathrm{Cik}(r'_{12}-h_1-l_1+ns_2) \qquad [7]$$
$$+\mathrm{Cik}(r'_{12}+h_1+l_1-ns_2)-\mathrm{Cik}(r'_{10}+h_1-ns_2)]ds_2$$

$$+\int_{0}^{l_2}\sin k[l_1+l_2+h_1-(n+1)s_2][\mathrm{Sik}(r'_{10}-h_1-ns_2)-\mathrm{Sik}(r'_{12}-h_1-l_1+ns_2) \qquad [8]$$
$$-\mathrm{Sik}(r'_{12}+h_1+l_1-ns_2)+\mathrm{Sik}(r'_{10}+h_1-ns_2)]ds_2$$

[continued on next sheet]

$$- j\Big\{-\int_{-l_2}^{0} \mathrm{Sin}\,k[l_1+l_2-h_1+(n+1)s_2][\mathrm{Ci}k(\acute{r}_{10}+h_1-ns_2) - \mathrm{Ci}k(\acute{r}_{11}+h_1-l_1-ns_2) \quad [9]$$
$$-\mathrm{Ci}k(\acute{r}_{11}-h_1+ns_2)+\mathrm{Ci}k(\acute{r}_{30}-h_1+ns_2)]ds_2$$

$$-\int_{l_2}^{0} \mathrm{Cos}\,k[l_1+l_2-h_1+(n+1)s_2][\mathrm{Si}k(\acute{r}_{10}+h_1-ns_2)-\mathrm{Si}k(\acute{r}_{11}+h_1-l_1-ns_2) \quad [10]$$
$$+\mathrm{Si}k(\acute{r}_{11}-h_1+ns_2)-\mathrm{Si}k(\acute{r}_{10}-h_1+ns_2)]ds_2$$

$$+\int_{-l_2}^{0} \sin k[l_1-l_2+h_1-(n+1)s_2][\mathrm{Ci}k(\acute{r}_{10}-h_1+ns_2)-\mathrm{Ci}k(\acute{r}_{12}-h_1-l_1+ns_2) \quad [11]$$
$$-\mathrm{Ci}k(\acute{r}_{12}+h_1+l_1-ns_2)+\mathrm{Ci}k(\acute{r}_{10}+h_1-ns_2)]ds_2$$

$$+\int_{-l_2}^{0} \cos k[l_1-l_2+h_1-(n+1)s_2][\mathrm{Si}k(\acute{r}_{10}-h_1+ns_2)-\mathrm{Si}k(\acute{r}_{12}-h_1-l_1+ns_2) \quad [12]$$
$$+\mathrm{Si}k(\acute{r}_{12}+h_1+l_1-ns_2)-\mathrm{Si}k(\acute{r}_{10}+h_1-ns_2)]ds_2 \qquad (8b)$$

$$+\int_{0}^{l_2} \sin k[l_1-l_2-h_1+(n+1)s_2][\mathrm{Ci}k(\acute{r}_{10}+h_1-ns_2)-\mathrm{Ci}k(\acute{r}_{10}+h_1-l_1-ns_2) \quad [13]$$
$$-\mathrm{Ci}k(\acute{r}_{11}-h_1+l_1+ns_2)+\mathrm{Ci}k(\acute{r}_{10}-h_1+ns_2)]ds_2$$

$$+\int_{0}^{l_2} \cos k[l_1-l_2-h_1+(n+1)s_2][\mathrm{Si}k(\acute{r}_{10}+h_1-ns_2)-\mathrm{Si}k(\acute{r}_{11}+h_1-l_1-ns_2) \quad [14]$$
$$+\mathrm{Si}k(\acute{r}_{11}-h_1+l_1+ns_2)-\mathrm{Si}k(\acute{r}_{10}-h_1+ns_2)]ds_2$$

$$-\int_{0}^{l_2} \sin k[l_1+l_2+h_1-(n+1)s_2][\mathrm{Ci}k(\acute{r}_{10}-h_1+ns_2)-\mathrm{Ci}k(\acute{r}_{12}-h_1-l_1+ns_2) \quad [15]$$
$$-\mathrm{Ci}k(\acute{r}_{12}+h_1+l_1-ns_2)+\mathrm{Ci}k(\acute{r}_{10}+h_1-ns_2)]ds_2$$

$$-\int_{0}^{l_2} \cos k[l_1+l_2+h_1-(n+1)s_2][\mathrm{Si}k(\acute{r}_{10}-h_1+ns_2)-\mathrm{Si}k(\acute{r}_{12}-h_1-l_1+ns_2) \quad [16]$$
$$+\mathrm{Si}k(\acute{r}_{12}+h_1+l_1-ns_2)-\mathrm{Si}k(\acute{r}_{10}+h_1-ns_2)]ds_2$$

3.12 In equation 3.11(8), let I_{2i} be the i^{th} integral $i=1\ldots16$, taken in the order written. Consequently,

$$\frac{2}{k(n+1)}(\acute{Z}_{21}+\acute{\acute{Z}}_{21}) = \sum_1^{16} I_{2i} \qquad (1)$$

Now consider each I_{2i} in succession, for example,

$$I_{21} = -\int_{-l_2}^{0} \cos k[l_1+l_2-h_1+(n+1)s_2][Cik(\bar{r}_{10}+h_1-ns_2)-Cik(\bar{r}_{11}+h_1-l_1-ns_2)$$
$$+Cik(\bar{r}_{11}-h_1+l_1+ns_2)-Cik(\bar{r}_{10}-h_1+ns_2)]ds_2 \quad (2)$$

Integrating by parts, let

$$u_{2i} = [Cik(\bar{r}_{10}+h_1-ns_2) \ldots -Cik(\bar{r}_{10}-h_1+ns_2)]$$

$$du_{2i} = [Ci'k(\bar{r}_{10}+h_1-ns_2) \frac{\partial}{\partial s_2} k(\bar{r}_{10}+h_1-ns_2) \ldots]\, ds_2$$

$$dv_{21} = -\cos k[l_1+l_2-h_1+(n+1)s_2 ds_2$$

$$v_{21} = -\frac{1}{k(n+1)} \sin k[l_1+l_2-h_1+(n+1)s_2]$$

and one obtains

$$I_{21} = -\frac{1}{k(n+1)} \sin k[l_1+l_2-h_1][Cik(\bar{r}_{2010}+h_1) - Cik(\bar{r}_{2011}+h_1-l_1)$$
$$+Cik(\bar{r}_{2011}-h_1+l_1) - Cik(\bar{r}_{2010}-h_1+ns_2) \quad (3)$$

$$+ \frac{1}{k(n+1)} \sin k[l_1-nl_2-h_1][Cik(\bar{r}_{2110}+h_1+nl_2)-Cik(\bar{r}_{2111}+h_1-l_1+nl_2)$$
$$+Cik(\bar{r}_{2111}-h_1+l_1-nl_2)-Cik(\bar{r}_{2110}-h_1-nl_2)]$$
$$+ \frac{1}{k(n+1)} \int_{l_2}^{0} \sin k[l_1+l_2-h_1+(n+1)s_2]du_{21}$$

$$= \frac{1}{k(n+1)} \sin k y_{21}[Cik\bar{\bar{L}}^2_{31}-Cik\bar{\bar{L}}^2_{61}+Cik\bar{\bar{L}}^2_{62}-Cik\bar{\bar{L}}^2_{32}]$$
$$- \frac{1}{k(n+1)} \sin k z_{26}[Cik\bar{\bar{L}}^2_{61}-Cik\bar{\bar{L}}^2_{81}+Cik\bar{\bar{L}}^2_{82}-Cik\bar{\bar{L}}^2_{82}] \quad (4)$$
$$+ \frac{1}{k(n+1)} \int_{-l_2}^{0} \sin k[l_1+l_2-h_1+(n+1)s_2]du_{21}$$

Comparing (4) with 3,9(1), it is seen that the sum of the uv terms yields

$$\frac{2}{k(n+1)} Z'_{21}$$

that is

$$\Sigma_1^{16} I_{2i} = \frac{2}{k(n+1)}[Z'_{21} - \frac{1}{2}\Sigma_1^{16} \int_{lim} v_{2i} du_{2i}]$$

or

$$Z''_{2i} = -\frac{1}{2}\Sigma_1^{16} \int_{lim} v_{2i} du_{2i} \quad (5)$$

Returning to equation 3.11(3), reversing the order of integration yields an expression for $\overset{,,}{Z}_{12}$ similar to (5), in which the subscripts and corresponding projections are interchanged, that is,

$$\overset{,,}{Z}_{12} = -\frac{1}{2}\Sigma_1^{16}\int_{lim} v_{1i}\,du_{1i} \qquad (6)$$

Substituting (5) and (6) into 3.11(3), and subsequently into 3.11(2):

$$\frac{\sin kl_1 \sin kl_2}{j30}\left(\bar{\bar{Z}}_{21}+\bar{\bar{Z}}_{12}\right) = \frac{n^2}{n+1}\left(\acute{Z}_{21}+\acute{Z}_{12}\right) + \frac{1}{n+1}\left(\overset{,,}{Z}_{21}+\overset{,,}{Z}_{12}\right) . \qquad (7)$$

Returning to (2), if the interval of integration were such that u_{21} were monotonic, one could apply the second law of the mean of the integral calculus and write

$$I_{21} = u_{21}v_{21}\Big]_{-l_2}^{0} - v_{21}(x_{21})\int_{-l_2}^{0} du_{21}$$

or

$$\overset{,,}{Z}_{21} = -\frac{1}{2}\Sigma_1^{16}\, v_{2i}(x_{2i})u_{2i}\Big]_{limits}$$

and

$$\overset{,,}{Z}_{12} = -\frac{1}{2}\Sigma_1^{16}\, v_{1i}(x_{1i})u_{1i}\Big]_{limits} .$$

Even though the law of the mean can not be applied directly, one can find numerous examples involving

$$\int_a^b \cos x\, f(x)\, dx$$

where integration demonstrates that in some cases it is legitimate to write

$$\int_a^b \cos x f(x) dx = \sin x f(x)\Big]_a^b \pm \sin x_\sigma \int_a^b df(x) .$$

Now, actually,

$$u_{2i} \equiv u_{2i}(n, s_2) .$$

Hence, write

$$I_{2i} = [u_{2i}v_{2i}]_{lim} + f(n)v_{2i}(x_{2i})[u_{2i}]_{lim} \tag{8}$$

in which x_{2i} and $f(n)$ remain to be determined. Thus, write

$$Z'_{21} = \tfrac{1}{2}f(n)\Sigma_1^{16} v_{2i}(x_{2i})[u_{2i}]_{lim} \ . \tag{9}$$

A study of 3.11(8), 3.12(4), 3.9(1,2), and (7) above, shows that

$$\text{for } n=1, \quad Z''_{21} = Z'_{21}$$

and that this will be true if the following conditions are imposed:

$$\begin{aligned} f(1) &= 1 \\ x_{2i} &= -l_2, \ i=1,2,3,4,9,10,11,12 \\ x_{2i} &= +l_2, \ i=5,6,7,8,13,14,15,16. \end{aligned} \tag{10}$$

However, (10) is not sufficient to determine $f(n)$ for $n \neq 1$.

3.13 It was pointed out in paragraph 2.3 that the electric field at point P has two components in cylindrical coordinates, namely

$$\begin{aligned} E_z &= j30I[2\cos kl_1\frac{e^{-jkr_0}}{r_0} - \frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_2}}{r_2}] \\ E_r &= -j30I[2\cos kl_1\frac{e^{-jkr_0}}{r_0}\cot\theta_0 - \frac{e^{-jkr_1}}{r_1}\cot\theta_1 - \frac{e^{-jkr_2}}{r_2}\cot\theta_2] \end{aligned} \tag{1}$$

Now, if $\bar{a}_1\cdot\bar{a}_2=0$ and either $\bar{r}_c\cdot\bar{a}_1=0$ or $\bar{r}_c\cdot\bar{a}_2=0$, or if $\bar{a}_1\cdot\bar{a}_2=0$ and both $\bar{r}_c\cdot\bar{a}_1=0$ and $\bar{r}_c\cdot\bar{a}_2=0$, examination of (1) shows that $Z_{21}=0$. Hence, for those terms in Z''_{21} which arise from integrations over the interval pairs

$$\begin{aligned} &(P_{10}P_{12}) \text{ and } (P_{20}P_{22}), \quad f(0) = 1, \\ &(P_{11}P_{10}) \text{ and } (P_{21}P_{20}), \quad f(0) = 1, \\ &(P_{10}P_{12}) \text{ and } (P_{21}P_{20}), \quad f(0) = -1, \\ &(P_{11}P_{10}) \text{ and } (P_{20}P_{22}), \quad f(0) = -1. \end{aligned} \tag{2}$$

As a further check, the choice of (2) leaves $Z'_{21} \neq 0$ for the case

where both $\bar{a}_1.\bar{a}_2 = o$ and $\bar{a}_1.\bar{a}_2 x\bar{a}_c = o$ but neither $\bar{a}_1.\bar{a}_c = o$ nor $\bar{a}_2.\bar{a}_c = o$. The choice of signs is further strengthened by consideration of the dominant terms in Z'_{21} as the antennas approach parallelism and normalcy, respectively.

As parallelism is approached, the im pedance is determined by $\bar{a}_1.\bar{E}_z$. On the other hand, as normalcy is approached, the impedance is determined by $\bar{a}_1.E_\rho$, and hence f(n) must also contain the terms $(\bar{a}_1.\bar{a}_\rho)^2$ to care for the contribution to the impedance by $\bar{a}_1.\bar{E}_\rho$. However, the signs of the terms of Z'_{21} to which the f ac tor $(\bar{a}_1.\bar{a}_\rho)^2$ is applied must be chosen in accordance with (2) above.

Having defined

$$\bar{a}_\rho = \bar{a}_2 x \frac{\bar{a}_c x \bar{a}_2}{|\bar{a}_c x \bar{a}_2|} \quad , \quad \bar{a}_2.\bar{a}_1 = \cos\theta \; ,$$

for coplanar antennas,

$$\bar{a}_\rho.\bar{a}_1 = \sin\theta \; .$$

Thus, if the antennas are coplanar, imposing all the above conditions upon Z'_{21}, and by analogy upon Z'_{12}, one finally obtains for those terms involving integration over the interval pairs

$(P_{10}P_{12})$ and $(P_{20}P_{22})$, $\quad f(n) = \cos^2\theta + \sin^2\theta = 1$

$(P_{11}P_{10})$ and $(P_{21}P_{20})$, $\quad f(n) = \cos^2\theta + \sin^2\theta = 1$

$(P_{10}P_{12})$ and $(P_{21}P_{20})$, $\quad f(n) = \cos^2\theta - \sin^2\theta = \cos 2\theta$

$(P_{11}P_{10})$ and $(P_{20}P_{22})$, $\quad f(n) = \cos^2\theta - \sin^2\theta = \cos 2\theta$

Combining (3) above with 3.12(10) and 3.12(4), in the case of coplanar antennas, the expression for Z'_{21} may be written ;

$$
\begin{aligned}
2Z''_{21} = \; & \\
& \sin kz_{24}[\mathrm{Cik}\bar{\bar{L}}^2_{21}-\mathrm{Cik}\bar{\bar{L}}^2_{41}+\mathrm{Cik}\bar{\bar{L}}^2_{42}-\mathrm{Cik}\bar{\bar{L}}^2_{22}-\mathrm{Cik}\bar{\bar{L}}^2_{51}+\mathrm{Cik}\bar{\bar{L}}^2_{71}-\mathrm{Cik}\bar{\bar{L}}^2_{72}+\mathrm{Cik}\bar{\bar{L}}^2_{52}] \\
& +\cos kz_{24}[\mathrm{Sik}\bar{\bar{L}}^2_{21}-\mathrm{Sik}\bar{\bar{L}}^2_{41}-\mathrm{Sik}\bar{\bar{L}}^2_{42}+\mathrm{Sik}\bar{\bar{L}}^2_{22}-\mathrm{Sik}\bar{\bar{L}}^2_{51}+\mathrm{Sik}\bar{\bar{L}}^2_{71}+\mathrm{Sik}\bar{\bar{L}}^2_{72}-\mathrm{Sik}\bar{\bar{L}}^2_{52}] \\
& +\sin kz_{29}[\mathrm{Cik}\bar{\bar{L}}^2_{81}-\mathrm{Cik}\bar{\bar{L}}^2_{91}+\mathrm{Cik}\bar{\bar{L}}^2_{92}-\mathrm{Cik}\bar{\bar{L}}^2_{82}-\mathrm{Cik}\bar{\bar{L}}^2_{51}+\mathrm{Cik}\bar{\bar{L}}^2_{71}-\mathrm{Cik}\bar{\bar{L}}^2_{72}+\mathrm{Cik}\bar{\bar{L}}^2_{52}]\cos 2\theta \\
& +\cos kz_{29}[\mathrm{Sik}\bar{\bar{L}}^2_{81}-\mathrm{Sik}\bar{\bar{L}}^2_{91}-\mathrm{Sik}\bar{\bar{L}}^2_{92}+\mathrm{Sik}\bar{\bar{L}}^2_{82}-\mathrm{Sik}\bar{\bar{L}}^2_{51}+\mathrm{Sik}\bar{\bar{L}}^2_{71}+\mathrm{Sik}\bar{\bar{L}}^2_{72}-\mathrm{Sik}\bar{\bar{L}}^2_{52}]\cos 2\theta \\
& -\sin kz_{21}[\mathrm{Cik}\bar{\bar{L}}^2_{11}-\mathrm{Cik}\bar{\bar{L}}^2_{21}+\mathrm{Cik}\bar{\bar{L}}^2_{22}-\mathrm{Cik}\bar{\bar{L}}^2_{12}-\mathrm{Cik}\bar{\bar{L}}^2_{31}+\mathrm{Cik}\bar{\bar{L}}^2_{61}-\mathrm{Cik}\bar{\bar{L}}^2_{62}+\mathrm{Cik}\bar{\bar{L}}^2_{32}]\cos 2\theta \\
& -\cos kz_{21}[\mathrm{Sik}\bar{\bar{L}}^2_{11}-\mathrm{Sik}\bar{\bar{L}}^2_{21}-\mathrm{Sik}\bar{\bar{L}}^2_{22}+\mathrm{Sik}\bar{\bar{L}}^2_{12}-\mathrm{Sik}\bar{\bar{L}}^2_{31}+\mathrm{Sik}\bar{\bar{L}}^2_{61}+\mathrm{Sik}\bar{\bar{L}}^2_{62}-\mathrm{Sik}\bar{\bar{L}}^2_{32}]\cos 2\theta \\
& -\sin kz_{26}[\mathrm{Cik}\bar{\bar{L}}^2_{61}-\mathrm{Cik}\bar{\bar{L}}^2_{81}+\mathrm{Cik}\bar{\bar{L}}^2_{82}-\mathrm{Cik}\bar{\bar{L}}^2_{62}-\mathrm{Cik}\bar{\bar{L}}^2_{31}+\mathrm{Cik}\bar{\bar{L}}^2_{51}-\mathrm{Cik}\bar{\bar{L}}^2_{52}+\mathrm{Cik}\bar{\bar{L}}^2_{32}] \\
& -\cos kz_{26}[\mathrm{Sik}\bar{\bar{L}}^2_{61}-\mathrm{Sik}\bar{\bar{L}}^2_{81}-\mathrm{Sik}\bar{\bar{L}}^2_{82}+\mathrm{Sik}\bar{\bar{L}}^2_{62}-\mathrm{Sik}\bar{\bar{L}}^2_{31}+\mathrm{Sik}\bar{\bar{L}}^2_{51}+\mathrm{Sik}\bar{\bar{L}}^2_{52}-\mathrm{Sik}\bar{\bar{L}}^2_{32}] \\
& -j\Big\{\sin kz_{24}[\mathrm{Sik}\bar{\bar{L}}^2_{21}-\mathrm{Sik}\bar{\bar{L}}^2_{41}+\mathrm{Sik}\bar{\bar{L}}^2_{42}-\mathrm{Sik}\bar{\bar{L}}^2_{22}-\mathrm{Sik}\bar{\bar{L}}^2_{51}+\mathrm{Sik}\bar{\bar{L}}^2_{71}-\mathrm{Sik}\bar{\bar{L}}^2_{72}+\mathrm{Sik}\bar{\bar{L}}^2_{52}] \qquad (5) \\
& \quad -\cos kz_{24}[\mathrm{Cik}\bar{\bar{L}}^2_{21}-\mathrm{Cik}\bar{\bar{L}}^2_{41}-\mathrm{Cik}\bar{\bar{L}}^2_{42}+\mathrm{Cik}\bar{\bar{L}}^2_{22}-\mathrm{Cik}\bar{\bar{L}}^2_{51}+\mathrm{Cik}\bar{\bar{L}}^2_{71}+\mathrm{Cik}\bar{\bar{L}}^2_{72}-\mathrm{Cik}\bar{\bar{L}}^2_{52}] \\
& \quad +\sin kz_{29}[\mathrm{Sik}\bar{\bar{L}}^2_{81}-\mathrm{Sik}\bar{\bar{L}}^2_{91}+\mathrm{Sik}\bar{\bar{L}}^2_{92}-\mathrm{Sik}\bar{\bar{L}}^2_{82}-\mathrm{Sik}\bar{\bar{L}}^2_{51}+\mathrm{Sik}\bar{\bar{L}}^2_{71}-\mathrm{Sik}\bar{\bar{L}}^2_{72}+\mathrm{Sik}\bar{\bar{L}}^2_{52}]\cos 2\theta \\
& \quad -\cos kz_{29}[\mathrm{Cik}\bar{\bar{L}}^2_{81}-\mathrm{Cik}\bar{\bar{L}}^2_{91}-\mathrm{Cik}\bar{\bar{L}}^2_{92}+\mathrm{Cik}\bar{\bar{L}}^2_{82}-\mathrm{Cik}\bar{\bar{L}}^2_{51}+\mathrm{Cik}\bar{\bar{L}}^2_{71}+\mathrm{Cik}\bar{\bar{L}}^2_{72}-\mathrm{Cik}\bar{\bar{L}}^2_{52}]\cos 2\theta \\
& \quad -\sin kz_{21}[\mathrm{Sik}\bar{\bar{L}}^2_{11}-\mathrm{Sik}\bar{\bar{L}}^2_{21}+\mathrm{Sik}\bar{\bar{L}}^2_{22}-\mathrm{Sik}\bar{\bar{L}}^2_{12}-\mathrm{Sik}\bar{\bar{L}}^2_{31}+\mathrm{Sik}\bar{\bar{L}}^2_{61}-\mathrm{Sik}\bar{\bar{L}}^2_{62}+\mathrm{Sik}\bar{\bar{L}}^2_{32}]\cos 2\theta \\
& \quad +\cos kz_{21}[\mathrm{Cik}\bar{\bar{L}}^2_{11}-\mathrm{Cik}\bar{\bar{L}}^2_{21}-\mathrm{Cik}\bar{\bar{L}}^2_{22}+\mathrm{Cik}\bar{\bar{L}}^2_{12}-\mathrm{Cik}\bar{\bar{L}}^2_{31}+\mathrm{Cik}\bar{\bar{L}}^2_{61}+\mathrm{Cik}\bar{\bar{L}}^2_{62}-\mathrm{Cik}\bar{\bar{L}}^2_{32}]\cos 2\theta \\
& \quad -\sin kz_{28}[\mathrm{Sik}\bar{\bar{L}}^2_{61}-\mathrm{Sik}\bar{\bar{L}}^2_{81}+\mathrm{Sik}\bar{\bar{L}}^2_{82}-\mathrm{Sik}\bar{\bar{L}}^2_{62}-\mathrm{Sik}\bar{\bar{L}}^2_{31}+\mathrm{Sik}\bar{\bar{L}}^2_{51}-\mathrm{Sik}\bar{\bar{L}}^2_{52}+\mathrm{Sik}\bar{\bar{L}}^2_{32}] \\
& \quad +\cos kz_{28}[\mathrm{Cik}\bar{\bar{L}}^2_{61}-\mathrm{Cik}\bar{\bar{L}}^2_{81}-\mathrm{Cik}\bar{\bar{L}}^2_{82}+\mathrm{Cik}\bar{\bar{L}}^2_{62}-\mathrm{Cik}\bar{\bar{L}}^2_{31}+\mathrm{Cik}\bar{\bar{L}}^2_{51}+\mathrm{Cik}\bar{\bar{L}}^2_{52}-\mathrm{Cik}\bar{\bar{L}}^2_{32}]\Big\}
\end{aligned}
$$

To write Z'_{12}, replace z_{2i} by z_{1i} and replace $\bar{\bar{L}}^2_{ij}$ by $\bar{\bar{L}}^1_{ij}$, in which h_1 is replaced by h_2, and in which l_1 and l_2 are interchanged. Thus, finally

$$
Z_{21} = Z_{12} = \frac{j15[(Z''_{21}+Z''_{12}) + (Z'_{21}+Z'_{12})\cos^2\theta]}{\sin kl_1 \, \sin kl_2 \, (1+\cos\theta)} \qquad (6)
$$

with

$$
0 \le \theta \le \frac{\pi}{2}, \quad \cos\theta = \bar{a}_1 \cdot \bar{a}_2, \quad h_1 = \bar{r}_0 \cdot \bar{a}_1, \quad h_2 = -\bar{r}_0 \cdot \bar{a}_2 .
$$

For non-coplanar antennas, let

$$\bar{a}_{2r} \equiv \bar{a}_2 \times \frac{(\bar{a}_c \times \bar{a}_2)}{|\bar{a}_c \times \bar{a}_2|} , \quad p_1 \equiv \bar{a}_1 \cdot \bar{a}_{2r} \equiv \sin\theta\cos\varphi ,$$

and Z_{21}'' becomes:

$$
\begin{aligned}
2Z_{21}'' = \; & \sin kz_{24}[\mathrm{Ci}k\bar{L}^2_{21}-\mathrm{Ci}k\bar{\bar{L}}^2_{41}+\mathrm{Ci}k\bar{L}^2_{42}-\mathrm{Ci}k\bar{L}^2_{22}-\mathrm{Ci}k\bar{\bar{L}}^2_{51}+\mathrm{Ci}k\bar{L}^2_{71}-\mathrm{Ci}k\bar{L}^2_{72}+\mathrm{Ci}k\bar{L}^2_{52}](n^2+p_1^2) \\
& +\cos kz_{24}[\mathrm{Si}k\bar{L}^2_{21}-\mathrm{Si}k\bar{L}^2_{41}-\mathrm{Si}k\bar{\bar{L}}^2_{42}+\mathrm{Si}k\bar{L}^2_{22}-\mathrm{Si}k\bar{L}^2_{51}+\mathrm{Si}k\bar{L}^2_{71}+\mathrm{Si}k\bar{L}^2_{72}-\mathrm{Si}k\bar{L}^2_{52}](n^2+p_1^2) \\
& +\sin kz_{29}[\mathrm{Ci}k\bar{L}^2_{81}-\mathrm{Ci}k\bar{\bar{L}}^2_{91}-\mathrm{Ci}k\bar{\bar{L}}^2_{82}-\mathrm{Ci}k\bar{L}^2_{51}+\mathrm{Ci}k\bar{L}^2_{71}-\mathrm{Ci}k\bar{L}^2_{72}+\mathrm{Ci}k\bar{\bar{L}}^2_{52}+\mathrm{Ci}k\bar{L}^2_{92}](n^2-p_1^2) \\
& +\cos kz_{29}[\mathrm{Si}k\bar{\bar{L}}^2_{81}-\mathrm{Si}k\bar{L}^2_{91}-\mathrm{Si}k\bar{L}^2_{92}+\mathrm{Si}k\bar{L}^2_{82}-\mathrm{Si}k\bar{L}^2_{51}+\mathrm{Si}k\bar{\bar{L}}^2_{71}+\mathrm{Si}k\bar{L}^2_{72}-\mathrm{Si}k\bar{\bar{L}}^2_{52}](n^2-p_1^2) \\
& -\sin kz_{21}[\mathrm{Ci}k\bar{\bar{L}}^2_{11}-\mathrm{Ci}k\bar{\bar{L}}^2_{21}+\mathrm{Ci}k\bar{\bar{L}}^2_{22}-\mathrm{Ci}k\bar{\bar{L}}^2_{12}-\mathrm{Ci}k\bar{\bar{L}}^2_{31}+\mathrm{Ci}k\bar{\bar{L}}^2_{51}-\mathrm{Ci}k\bar{L}^2_{52}+\mathrm{Ci}k\bar{L}^2_{32}](n^2-p_1^2) \\
& -\cos kz_{21}[\mathrm{Si}k\bar{\bar{L}}^2_{11}-\mathrm{Si}k\bar{\bar{L}}^2_{21}-\mathrm{Si}k\bar{\bar{L}}^2_{22}+\mathrm{Si}k\bar{L}^2_{12}-\mathrm{Si}k\bar{\bar{L}}^2_{31}+\mathrm{Si}k\bar{\bar{L}}^2_{51}+\mathrm{Si}k\bar{L}^2_{52}-\mathrm{Si}k\bar{\bar{L}}^2_{32}](n^2-p_1^2) \\
& -\sin kz_{26}[\mathrm{Ci}k\bar{L}^2_{61}-\mathrm{Ci}k\bar{L}^2_{81}+\mathrm{Ci}k\bar{L}^2_{82}-\mathrm{Ci}k\bar{L}^2_{62}-\mathrm{Ci}k\bar{L}^2_{31}+\mathrm{Ci}k\bar{L}^2_{51}-\mathrm{Ci}k\bar{L}^2_{52}+\mathrm{Ci}k\bar{\bar{L}}^2_{32}](n^2+p_1^2) \\
& -\cos kz_{26}[\mathrm{Si}k\bar{\bar{L}}^2_{61}-\mathrm{Si}k\bar{L}^2_{81}-\mathrm{Si}k\bar{\bar{L}}^2_{82}+\mathrm{Si}k\bar{L}^2_{62}-\mathrm{Si}k\bar{\bar{L}}^2_{31}+\mathrm{Si}k\bar{L}^2_{51}+\mathrm{Si}k\bar{L}^2_{52}-\mathrm{Si}k\bar{\bar{L}}^2_{32}](n^2+p_1^2) \\
& -j\Big\{\sin kz_{24}[\mathrm{Si}k\bar{L}^2_{21}-\mathrm{Si}k\bar{\bar{L}}^2_{41}+\mathrm{Si}k\bar{L}^2_{42}-\mathrm{Si}k\bar{L}^2_{22}+\mathrm{Si}k\bar{L}^2_{71}-\mathrm{Si}k\bar{\bar{L}}^2_{72}+\mathrm{Si}k\bar{\bar{L}}^2_{52}-\mathrm{Si}k\bar{L}^2_{51}](n^2+p_1^2) \\
& \quad -\cos kz_{24}[\mathrm{Ci}k\bar{L}^2_{21}-\mathrm{Ci}k\bar{\bar{L}}^2_{41}-\mathrm{Ci}k\bar{L}^2_{42}+\mathrm{Ci}k\bar{L}^2_{22}-\mathrm{Ci}k\bar{L}^2_{51}+\mathrm{Ci}k\bar{\bar{L}}^2_{71}+\mathrm{Ci}k\bar{L}^2_{72}-\mathrm{Si}k\bar{L}^2_{52}](n^2+p_1^2) \\
& \quad +\sin kz_{29}[\mathrm{Si}k\bar{L}^2_{81}-\mathrm{Si}k\bar{\bar{L}}^2_{91}+\mathrm{Si}k\bar{L}^2_{92}-\mathrm{Si}k\bar{\bar{L}}^2_{82}-\mathrm{Si}k\bar{\bar{L}}^2_{51}+\mathrm{Si}k\bar{\bar{L}}^2_{71}-\mathrm{Si}k\bar{L}^2_{72}+\mathrm{Si}k\bar{\bar{L}}^2_{52}](n^2-p_1^2) \\
& \quad -\cos kz_{29}[\mathrm{Ci}k\bar{L}^2_{81}-\mathrm{Ci}k\bar{L}^2_{91}-\mathrm{Ci}k\bar{\bar{L}}^2_{92}+\mathrm{Ci}k\bar{L}^2_{82}-\mathrm{Ci}k\bar{\bar{L}}^2_{51}+\mathrm{Ci}k\bar{L}^2_{72}+\mathrm{Ci}k\bar{\bar{L}}^2_{71}-\mathrm{Ci}k\bar{L}^2_{52}](n^2-p_1^2) \\
& \quad -\sin kz_{21}[\mathrm{Si}k\bar{\bar{L}}^2_{11}-\mathrm{Si}k\bar{L}^2_{21}+\mathrm{Si}k\bar{L}^2_{22}-\mathrm{Si}k\bar{\bar{L}}^2_{12}-\mathrm{Si}k\bar{L}^2_{31}+\mathrm{Si}k\bar{L}^2_{51}-\mathrm{Si}k\bar{L}^2_{52}+\mathrm{Si}k\bar{L}^2_{32}](n^2-p_1^2) \\
& \quad +\cos kz_{21}[\mathrm{Ci}k\bar{L}^2_{11}-\mathrm{Ci}k\bar{L}^2_{21}-\mathrm{Ci}k\bar{L}^2_{22}+\mathrm{Ci}k\bar{\bar{L}}^2_{12}-\mathrm{Ci}k\bar{L}^2_{31}+\mathrm{Ci}k\bar{L}^2_{81}+\mathrm{Ci}k\bar{\bar{L}}^2_{52}-\mathrm{Ci}k\bar{L}^2_{32}](n^2-p_1^2) \\
& \quad -\sin kz_{26}[\mathrm{Si}k\bar{L}^2_{61}-\mathrm{Si}k\bar{L}^2_{81}+\mathrm{Si}k\bar{L}^2_{82}-\mathrm{Si}k\bar{L}^2_{62}-\mathrm{Si}k\bar{L}^2_{31}+\mathrm{Si}k\bar{L}^2_{61}-\mathrm{Si}k\bar{L}^2_{52}+\mathrm{Si}k\bar{\bar{L}}^2_{32}](n^2+p_1^2) \\
& \quad +\cos kz_{26}[\mathrm{Ci}k\bar{L}^2_{61}-\mathrm{Ci}k\bar{L}^2_{81}-\mathrm{Ci}k\bar{\bar{L}}^2_{82}+\mathrm{Ci}k\bar{\bar{L}}^2_{62}-\mathrm{Ci}k\bar{\bar{L}}^2_{31}+\mathrm{Ci}k\bar{\bar{L}}^2_{61}+\mathrm{Ci}k\bar{\bar{L}}^2_{52}-\mathrm{Ci}k\bar{\bar{L}}^2_{32}](n^2+p_1^2)\Big\}
\end{aligned}
\tag{7}
$$

A similar expression may be written for Z_{12}'' and finally Z_{21} may be written from (6).

3.14 If a closer inspection of the selection of those terms to which the factor $\cos 2\theta$ is to be applied is thought necessary, return to 3.2(e) and write the last term:

$$Z_{12}^{1} = \frac{j30}{k\sin kl_1 \sin kl_2}\oint_{P_{20}}^{P_{22}} \sin k(l_2-s_2)\left\{\left[\oint_{P_{11}}^{P_{10}} \sin k(l_1+s_1) + \oint_{P_{10}}^{P_{12}} \sin k(l_1-s_1)\right]\bar{\bar{\Delta}}_1\left[\frac{e^{-jkr_{21}}}{r_{21}} d\bar{s}_2\right]\right\}\cdot d\bar{s}_1 \qquad (1)$$

Now, integration of the terms within the brace will give the negative of the field from antenna one, which must be integrated against the current in the upper half of antenna two, that is, Z_{12}^{1} is the mutual impedance between antenna one and the upper half of antenna two. In other words, if the antennas were parallel with each normal to the line of centers, Z_{12}^{1} would be equivalent to the mutual impedance of two base fed vertical antennas above a perfect ground. However, in general, within the interval $(P_{11}P_{10})$, the vector $-\bar{s}_1$ would have to be replaced by the mirror image of vector $\bar{s}_1$ within $(P_{10}P_{12})$.

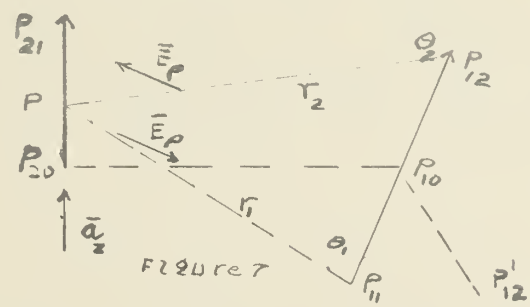


Figure 7

Returning to (1) and 3.13(1), upon examining the figure, it is seen that for the term involving $\cot\theta_1$,

$$\bar{a}_{r_1}\cdot\bar{a}_z < 0$$

and for those terms involving $\cot\theta_2$,

$$\bar{a}_{r_2}\cdot\bar{a}_z > 0 \quad ,$$

thus verifying that the contribution from the radial field to Z_{12} has the same sign as that from the axial field for the interval $(P_{10}P_{12})$ and has the opposite sign for the interval $(P_{11}P_{10})$.

A similar examination of

$$Z'_{12} = \frac{j30}{k\sin kl_1 \sin kl_2}\oint_{P_{10}}^{P_{12}} \sin k(l_1-s_1)\Big\{\Big[\oint_{P_{21}}^{P_{20}} \sin k(l_2+s_2) + \oint_{P_{20}}^{P_{22}} \sin k(l_2-s_2)\Big]\bar{\bar{\Delta}}_2\Big[\frac{e^{-jkr_{12}}}{r_{12}}d\bar{s}_1\Big]\Big\}\cdot d\bar{s}_2 \tag{2}$$

yields the same result for $(P_{21}P_{20})$ and $(P_{20}P_{22})$. Hence, 3.13(4) is verified.

<u>3.15</u> Consider the special case where $\bar{a}_1\cdot\bar{a}_2 = 0$, $\bar{a}_1\cdot\bar{a}_2\times\bar{a}_c = 0$, $\bar{a}_1\cdot\bar{a}_c \neq 0$ and $\bar{a}_2\cdot\bar{a}_c \neq 0$.

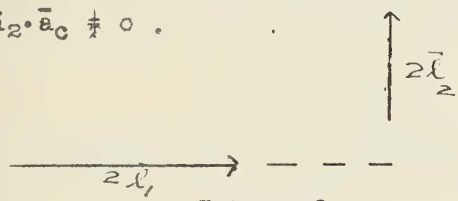


Figure 8

Here,

$$\theta = \frac{\pi}{2}, \quad n = 0, \quad \cos 2\theta = -1, \quad h_1 \neq 0,$$

$$z_{21} = z_{23} = z_{26}, \quad z_{22} = z_{25} = z_{28}, \quad z_{24} = z_{27} = z_{29},$$

$$z_{21} = h_1 - l_1, \quad z_{25} = h_1, \quad z_{24} = h_1 + l_1 .$$

Hence,

$$\bar{\bar{Z}}_{21}\frac{\sin kl_1 \sin kl_2}{15} =$$

$$\begin{aligned}\Big\{&-\cos kz_{21}[Cik\bar{\bar{L}}^2_{11}+Cik\bar{\bar{L}}^2_{12}-Cik\bar{\bar{L}}^2_{21}-Cik\bar{\bar{L}}^2_{22}-Cik\bar{\bar{L}}^2_{51}-Cik\bar{\bar{L}}^2_{52}+Cik\bar{\bar{L}}^2_{81}+Cik\bar{\bar{L}}^2_{82}]\\
&+\sin kz_{21}[Sik\bar{\bar{L}}^2_{11}-Sik\bar{\bar{L}}^2_{12}-Sik\bar{\bar{L}}^2_{21}+Sik\bar{\bar{L}}^2_{22}-Sik\bar{\bar{L}}^2_{51}+Sik\bar{\bar{L}}^2_{52}+Sik\bar{\bar{L}}^2_{81}-Sik\bar{\bar{L}}^2_{82}]\\
&-\cos kz_{24}[Cik\bar{\bar{L}}^2_{21}+Cik\bar{\bar{L}}^2_{22}-Cik\bar{\bar{L}}^2_{41}-Cik\bar{\bar{L}}^2_{42}-Cik\bar{\bar{L}}^2_{81}-Cik\bar{\bar{L}}^2_{82}+Cik\bar{\bar{L}}^2_{91}+Cik\bar{\bar{L}}^2_{82}]\\
&+\sin kz_{24}[Sik\bar{\bar{L}}^2_{21}-Sik\bar{\bar{L}}^2_{22}-Sik\bar{\bar{L}}^2_{41}+Sik\bar{\bar{L}}^2_{42}-Sik\bar{\bar{L}}^2_{81}+Sik\bar{\bar{L}}^2_{82}+Sik\bar{\bar{L}}^2_{91}-Sik\bar{\bar{L}}^2_{92}]\Big\}\end{aligned} \tag{1a}$$

[continued on next sheet]

$$
\begin{aligned}
+j\Big\{ & \sin kz_{21}[\mathrm{Cik}\bar{\bar{L}}^2_{11}-\mathrm{Cik}\bar{\bar{L}}^2_{12}-\mathrm{Cik}\bar{\bar{L}}^2_{21}+\mathrm{Cik}\bar{\bar{L}}^2_{22}-\mathrm{Cik}\bar{\bar{L}}^2_{81}+\mathrm{Cik}\bar{\bar{L}}^2_{82}+\mathrm{Cik}\bar{\bar{L}}^2_{81}-\mathrm{Cik}\bar{\bar{L}}^2_{82}] \\
& +\cos kz_{21}[\mathrm{Sik}\bar{\bar{L}}^2_{11}+\mathrm{Sik}\bar{\bar{L}}^2_{12}-\mathrm{Sik}\bar{\bar{L}}^2_{21}-\mathrm{Sik}\bar{\bar{L}}^2_{22}-\mathrm{Sik}\bar{\bar{L}}^2_{81}-\mathrm{Sik}\bar{\bar{L}}^2_{82}+\mathrm{Sik}\bar{\bar{L}}^2_{81}+\mathrm{Sik}\bar{\bar{L}}^2_{82}] \\
& +\sin kz_{24}[\mathrm{Cik}\bar{\bar{L}}^2_{21}-\mathrm{Cik}\bar{\bar{L}}^2_{22}-\mathrm{Cik}\bar{\bar{L}}^2_{41}+\mathrm{Cik}\bar{\bar{L}}^2_{42}-\mathrm{Cik}\bar{\bar{L}}^2_{81}+\mathrm{Cik}\bar{\bar{L}}^2_{82}+\mathrm{Cik}\bar{\bar{L}}^2_{91}-\mathrm{Cik}\bar{\bar{L}}^2_{92}] \\
& +\cos kz_{24}[\mathrm{Sik}\bar{\bar{L}}^2_{21}+\mathrm{Sik}\bar{\bar{L}}^2_{22}-\mathrm{Sik}\bar{\bar{L}}^2_{41}-\mathrm{Sik}\bar{\bar{L}}^2_{42}-\mathrm{Sik}\bar{\bar{L}}^2_{81}-\mathrm{Sik}\bar{\bar{L}}^2_{82}+\mathrm{Sik}\bar{\bar{L}}^2_{91}+\mathrm{Sik}\bar{\bar{L}}^2_{92}]\Big\}
\end{aligned}
\qquad (1b)
$$

with a similar expression for $\bar{\bar{Z}}_{12}$. Finally,

$$Z_{12} = Z_{12} = \tfrac{1}{2}(\bar{\bar{Z}}_{12} + \bar{\bar{Z}}_{21}) .$$

In the above case, if $\bar{a}_1 \cdot \bar{a}_c = h_1 = o$,

$$z_{21} = -l_1, \quad z_{24} = l_1, \quad \bar{\bar{L}}^2_{41} = \bar{\bar{L}}^2_{11}, \quad \text{etc.},$$

and hence

$$Z_{21} = o .$$

Also, if $\bar{a}_1 \cdot \bar{a}_2 = o$, and $\bar{a}_1 \cdot \bar{a}_2 x \bar{a}_c = 1$,

$$\bar{\bar{L}}^2_{11} = \bar{\bar{L}}^2_{61}, \quad \bar{\bar{L}}^2_{21} = \bar{\bar{L}}^2_{81}, \quad \bar{\bar{L}}^2_{41} = \bar{\bar{L}}^2_{91}, \quad \text{etc., and} \quad \bar{a}_r \cdot \bar{x}_1 = o.$$

Hence, again

$$Z_{21} = o .$$

The above cases check with the conditions imposed upon Z_{21} when determining $f(n)$.

3.16 Consider a biconical antenna whose radius at the feed point is b and whose radius at the ends is a, with

$$b < a << l_1$$

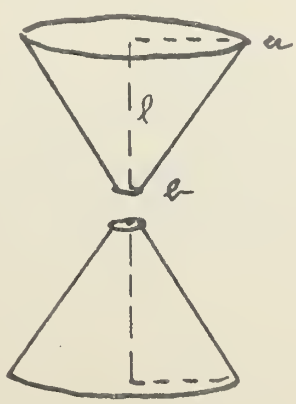


Figure 9

In this case,

$$
\begin{aligned}
&h_1 = 0 \\
&-z_{21} = z_{29} = (l_1 + nl_2) = 2l \\
&-z_{22} = z_{28} = nl_2 = l \\
&-z_{23} = z_{27} = l_1 \\
&-z_{26} = z_{24} = l_1 - nl_2 \doteq 0 \\
&z_{25} = 0 \\
&\bar{r}_3 = \bar{r}_7 = \sqrt{l_1^2 + b^2} \doteq l + \frac{b^2}{2l} \\
&\bar{r}_2 = \bar{r}_8 = \sqrt{l_1^2 + a^2} \doteq l + \frac{a^2}{2l} \qquad\qquad (1) \\
&\bar{r}_1 = \bar{r}_9 = \sqrt{4l_1^2 + a^2} \doteq 2l + \frac{a^2}{4l} \\
&\bar{r}_4 = \bar{r}_6 = a, \quad \bar{r}_5 = b \\
&\bar{\bar{L}}^2_{21} = \bar{\bar{L}}^2_{82}, \quad \bar{\bar{L}}^2_{31} = \bar{\bar{L}}^2_{72}, \quad \bar{\bar{L}}^2_{32} = \bar{\bar{L}}^2_{71}, \quad \bar{\bar{L}}^2_{22} = \bar{\bar{L}}^2_{81}, \\
&\bar{\bar{L}}^2_{41} = \bar{\bar{L}}^2_{42} = \bar{\bar{L}}^2_{61} = \bar{\bar{L}}^2_{62} .
\end{aligned}
$$

For small arguments,

$$\mathrm{Ci}x \doteq C + \ln x \quad \text{and} \quad \mathrm{Si}x \doteq x,$$

with

$$C = 0.5772 \ldots$$

Hence, the input impedance of the antenna under the hypothesis of a sinusoidal current distribution may be written:

$$
\begin{aligned}
Z_{in}\sin^2 kl = \Big\{ & 60[C+\ln 2kl-\mathrm{Ci}2kl] - 30kl\frac{n-1}{n+1}\mathrm{Si}2kl \\
& +30\sin 2kl[\mathrm{Si}4kl-2\mathrm{Si}2kl]+30\cos 2kl[C+\ln kl-2\mathrm{Ci}2kl+\mathrm{Ci}4kl]\Big\} \qquad (2)\\
& +j\Big[60\mathrm{Si}2kl+30kl\frac{n-1}{n+1}[C+\ln\frac{kb^2}{2l}-\mathrm{Ci}2kl] \\
& \quad +30\sin 2kl[\mathrm{Ci}4kl-2\mathrm{Ci}2kl+\ln\frac{kb^2}{l}+C]+30\cos 2kl[2\mathrm{Si}2kl-\mathrm{Si}4kl]\Big\} .
\end{aligned}
$$

3.17 For small diameter straight wire antennas [illegible] be assumed sinusoidally distributed, letting [illegible]

$$
\begin{aligned}
\sin^2 kl Z_{in} = {} & 60(C+\ln 2kl-Ci\,2kl)+30(Si\,4kl-2Si\,2kl)\,[illegible] \\
& +30(C+\ln kl-2Ci\,2kl+Ci\,4kl)\cos 2kl \\
& +j\{60Si\,2kl+30(2Si\,2kl-Si\,4kl)\cos 2kl \\
& -30(\ln\frac{\lambda l}{a^2}-[illegible]-\ln 2\pi-Ci\,4kl+2Ci\,2kl)\sin 2kl\}
\end{aligned}
\qquad (1)
$$

Formula (1) is identical with that given in [illegible] texts[1] for the input impedance of thin wire antennas as developed by the induced E.M.F. method except it is in terms of the current at the center of a symmetrically fed antenna instead of in terms of the [illegible] maximum. It is usually derived by first considering the case of [illegible] antennas and then taking the mutual impedance between the axis and [illegible] element on the axis of the wire. If one wishes to include the ohmic loss, the term

$$R_1 = \frac{R_s\,[illegible]}{[illegible]}(1 - \frac{1}{kl}[illegible] + \cot^2 kl) \qquad (2)$$

with R_s being the surface resistivity of the wire, may be added.

3.18 The overall procedure for determining the mutual impedance between two straight, symmetrically fed, open wire antennas of relatively small cross section and in the neighborhood of a [illegible] may be summarized.

(1) Measure or compute the nine diagonal lengths.

(2) Measure the angle between the two antenna directions, choosing the positive senses such that this angle does not exceed a right angle.

(3) Measure the angle between each antenna [illegible] centers.

(4) Determine the projection of [illegible] the other.

(5) Lay off the y's and z's as in 3.8(13).

(6) For parallel-staggered antennas, the diagonals may be drawn to scale as in 3.8(13) and swung in [illegible] positive and negative directions from the determinating ordinate for determining the L's.

(7) For non-parallel antennas, the diagonals may be picked off as radii and swung as in (6) above from the correspondingly numbered z's.

(8) Having determined the lengths, convert to electrical lengths and determine Z''_{12} and Z''_{21} from 3.13(7), and determine Z'_{12} and Z'_{21} from 3.9(1). For parallel-staggered antennas, determine only Z'_{21}.

(9) Finally, write Z_{21} from either 3.13(8) or 3.9(2), depending upon whether the antennas have non-parallel directions or have parallel directions.

<u>3.18</u> A word regarding the rigor of formula 3.13(8) seems to be in order. The final integrations for Z''_{12} and Z''_{21} really constitute a form of extended curve fitting in which Z_{21} is caused to reduce to all special cases where the integrations can be carried out rigorously, and not to vanish in certain cases where it is known from the solutions for the fields that $Z_{21} \neq 0$. [illegible] confidence in the formula [illegible]

by maintaining physical symmetry in l_1 and l_2. Although it can not be definitely concluded, it is altogether likely that either

$$Z_{21} = \overset{\equiv}{Z}_{21} \quad \text{or} \quad Z_{21} = \overset{\equiv}{Z}_{12}$$

will give sufficiently accurate results.

Apparently, it was problems of this nature which many years ago caused E. B. Wilson to state that, in mathematical physics, one should prosecute his mathematics vigorously rather than rigorously.

IV

SPECIAL CASES

4.1 It was pointed out in 3.9(2) that in the case of parallel-staggered antennas, Z_{21} could be determined from those terms which were rigorously integrated. In this case, n = 1 and

$$z_{2i} = y_{2i} = z_{1i} = y_{1i} \equiv y_i \;, \; i = 1, \dots 9 \;. \qquad (1)$$

Thus, letting

$$Z_{21} = R_{21} + jX_{21}$$

one obtains from 3.9(1,2) :

$$R_{21}\frac{\sin kl_1 \sin kl_2}{15} =$$

$$\cos ky_1[CikL_{11}+CikL_{12}-CikL_{21}-CikL_{22}-CikL_{31}-CikL_{32}+CikL_{51}+CikL_{52}]$$
$$-\cos ky_4[CikL_{21}+CikL_{22}-CikL_{41}-CikL_{42}-CikL_{51}-CikL_{52}+CikL_{71}+CikL_{72}]$$
$$-\cos ky_6[CikL_{31}+CikL_{32}-CikL_{51}-CikL_{52}-CikL_{62}-CikL_{62}+CikL_{81}+CikL_{82}]$$
$$+\cos ky_9[CikL_{51}+CikL_{52}-CikL_{71}-CikL_{72}-CikL_{81}-CikL_{82}+CikL_{91}+CikL_{92}]$$
$$-\sin ky_1[SikL_{11}-SikL_{12}-SikL_{21}+SikL_{22}-SikL_{31}+SikL_{32}+SikL_{51}-SikL_{52}]$$
$$+\sin ky_4[SikL_{21}-SikL_{22}-SikL_{41}+SikL_{42}-SikL_{51}+SikL_{52}+SikL_{71}-SikL_{72}]$$
$$+\sin ky_6[SikL_{31}-SikL_{32}-SikL_{51}+SikL_{52}-SikL_{61}+SikL_{62}+SikL_{81}-SikL_{82}]$$
$$-\sin ky_9[SikL_{51}-SikL_{52}-SikL_{71}+SikL_{72}-SikL_{81}+SikL_{82}+SikL_{92}-SikL_{92}]$$

$$X_{21}\frac{\sin kl_1 \sin kl_2}{15} = \qquad (2)$$

$$-\sin ky_1[CikL_{11}-CikL_{12}-CikL_{21}+CikL_{22}-CikL_{31}+CikL_{32}+CikL_{51}-CikL_{52}]$$
$$+\sin ky_4[CikL_{21}-CikL_{22}-CikL_{41}+CikL_{42}-CikL_{51}+CikL_{52}+CikL_{71}-CikL_{72}]$$
$$+\sin ky_6[CikL_{31}-CikL_{32}-CikL_{51}+CikL_{52}-CikL_{62}+CikL_{62}+CikL_{81}-CikL_{82}]$$
$$-\sin ky_9[CikL_{51}-CikL_{52}-CikL_{71}+CikL_{72}-CikL_{81}+CikL_{82}+CikL_{92}-CikL_{92}]$$
$$-\cos ky_1[SikL_{11}+SikL_{12}-SikL_{21}-SikL_{22}-SikL_{31}-SikL_{32}+SikL_{51}+SikL_{52}]$$
$$+\cos ky_4[SikL_{21}+SikL_{22}-SikL_{41}-SikL_{42}-SikL_{51}-SikL_{52}+SikL_{71}+SikL_{72}]$$
$$+\cos ky_6[SikL_{31}+SikL_{32}-SikL_{51}-SikL_{52}-SikL_{61}-SikL_{62}+SikL_{81}+SikL_{82}]$$
$$-\cos ky_9[SikL_{51}+SikL_{52}-SikL_{71}-SikL_{72}-SikL_{81}-SikL_{82}+SikL_{91}+SikL_{92}]$$

with

$$y_1 = h-l_2-l_1,\ y_2=h-l_2,\ y_3=h-l_1,\ y_4=h-l_2+l_1,\ y_5 = h$$

$$y_9 = h+l_2+l_1,\ y_8=h+l_2,\ y_7=h+l_1,\ y_6=h+l_2-l_1 \qquad (3)$$

$$r_i = \sqrt{r^2+y_i^2}\ ,\ L_{i1} = r_i - y_i\ ,\ L_{i2} = r_i+y_i\ ,$$

r being the distance between the axes of the antennas and h being the the lateral displacements of the centers.

4.2 For parallel antennas not necessarily of the same length with $h=o$,

$$-y_1=y_9,\ -y_2=y_8,\ -y_3=y_7,\ -y_4=y_6,\ y_5=o,$$
$$r_1=r_9,\ r_2=r_8,\ r_3=r_7,\ r_4=r_6,\ r_5 = r = L_{51} = L_{52}. \qquad (1)$$

Hence

$$R_{21}\frac{\sin kl_1 \sin kl_2}{30} =$$
$$[2\mathrm{Ci}kr-\mathrm{Ci}kL_{71}-\mathrm{Ci}kL_{72}-\mathrm{Ci}kL_{81}-\mathrm{Ci}kL_{82}+\mathrm{Ci}kL_{91}+\mathrm{Ci}kL_{92}]\cos ky_9$$
$$-[\mathrm{Ci}kL_{71}+\mathrm{Ci}kL_{72}-2\mathrm{Ci}kr-\mathrm{Ci}kL_{61}-\mathrm{Ci}kL_{62}+\mathrm{Ci}kL_{81}+\mathrm{Ci}kL_{82}]\cos ky_6 \qquad (2a)$$
$$+[\mathrm{Si}kL_{71}-\mathrm{Si}kL_{72}+\mathrm{Si}kL_{81}-\mathrm{Si}kL_{82}-\mathrm{Si}kL_{91}+\mathrm{Si}kL_{92}]\sin ky_9$$
$$+[\mathrm{Si}kL_{72}-\mathrm{Si}kL_{71}-\mathrm{Si}kL_{61}+\mathrm{Si}kL_{62}+\mathrm{Si}kL_{81}-\mathrm{Si}kL_{82}]\sin ky_6$$

$$X_{21}\frac{\sin kl_1 \sin kl_2}{30} =$$
$$[\mathrm{Ci}kL_{71}-\mathrm{Ci}kL_{72}+\mathrm{Ci}kL_{81}-\mathrm{Ci}kL_{82}-\mathrm{Ci}kL_{91}+\mathrm{Ci}kL_{92}]\sin ky_9$$
$$+[\mathrm{Ci}kL_{72}-\mathrm{Ci}kL_{71}-\mathrm{Ci}kL_{61}+\mathrm{Ci}kL_{62}+\mathrm{Ci}kL_{81}-\mathrm{Ci}kL_{82}]\sin ky_6 \qquad (2b)$$
$$-[2\mathrm{Si}kr-\mathrm{Si}kL_{71}-\mathrm{Si}kL_{72}-\mathrm{Si}kL_{81}-\mathrm{Si}kL_{82}+\mathrm{Si}kL_{91}+\mathrm{Si}kL_{92}]\cos ky_9$$
$$+[-2\mathrm{Si}kr+\mathrm{Si}kL_{71}+\mathrm{Si}kL_{72}-\mathrm{Si}kL_{61}-\mathrm{Si}kL_{62}+\mathrm{Si}kL_{81}+\mathrm{Si}kL_{82}]\cos ky_6$$

with

$$y_9 = l_2 + l_1,\ y_6 = l_2 - l_1,\ k = \frac{2\pi}{\lambda}\ , \qquad (3)$$

and with

$$
\begin{aligned}
L_{61} &= \sqrt{r^2+(l_2-l_1)^2}-l_2+l_1, & L_{62} &= \sqrt{r^2+(l_2-l_1)^2}+l_2-l_1,\\
L_{71} &= \sqrt{r^2+l_1^2}-l_1, & L_{72} &= \sqrt{r^2+l_1^2}+l_1,\\
L_{81} &= \sqrt{r^2+l_2^2}-l_2, & L_{82} &= \sqrt{r^2+l_2^2}+l_2,\\
L_{91} &= \sqrt{r^2+(l_2+l_1)^2}-l_2-l_1, & L_{92} &= \sqrt{r^2+(l_1+l_2)^2}+l_2+l_1.
\end{aligned}
\tag{4}
$$

Formula (2) above is applicable to the solution for the input impedance of Yagi arrays(63). The mutual impedances may be readily determined by laying off the ordinates and diagonals to scale as in 3.6(13), converting the lengths into electrical lengths, and referring to tables listing the sine-integral and cosine-integral functions[62]. An example applying formula (2) to the analysis of such an antenna system will be given in ChapterVI.

4.5 An additional check upon formula 4.2(2), which formula has been previously reported without showing the derivation(63), may be obtained by letting $l_1=l_2$. Thus, for parallel antennas of equal length and normal to their line of centers,

$$
\begin{aligned}
R_{21} = {} & \frac{60}{\sin^2 kl}\left[2\mathrm{Ci}\,kr-\mathrm{Ci}\,kL_{71}-\mathrm{Ci}\,kL_{72}\right]\\
& +\frac{30}{\sin^2 kl}\left[2\mathrm{Ci}\,kr-2\mathrm{Ci}\,kL_{71}-2\mathrm{Ci}\,kL_{72}+\mathrm{Ci}\,kL_{91}+\mathrm{Ci}\,kL_{92}\right]\cos ky_9\\
& +\frac{30}{\sin^2 kl}\left[2\mathrm{Si}\,kL_{71}-2\mathrm{Si}\,kL_{72}-\mathrm{Si}\,kL_{91}+\mathrm{Si}\,kL_{92}\right]\sin ky_9
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
X_{21} = {} & \frac{60}{\sin^2 kl}\left[\mathrm{Si}\,kL_{71}+\mathrm{Si}\,kL_{72}-2\mathrm{Si}\,kr\right]\\
& +\frac{30}{\sin^2 kl}\left[2\mathrm{Si}\,kL_{71}+2\mathrm{Si}\,kL_{72}-2\mathrm{Si}\,kr-\mathrm{Si}\,kL_{91}-\mathrm{Si}\,kL_{92}\right]\cos ky_9\\
& +\frac{30}{\sin^2 kl}\left[2\mathrm{Ci}\,kL_{71}-2\mathrm{Ci}\,kL_{72}-\mathrm{Ci}\,kL_{91}+\mathrm{Ci}\,kL_{92}\right]
\end{aligned}
$$

with

$$l_1 = l_2 = l, \qquad y_9 = 2l$$

$$L_{71} = \sqrt{r^2 + l^2} - l, \quad L_{72} = \sqrt{r^2 + l^2} + l, \tag{2}$$

$$L_{91} = \sqrt{r^2 + 4l^2} - 2l, \quad L_{92} = \sqrt{r^2 + 4l^2} + 2l.$$

Formula (1) is identical with that found in the standard texts(1), and as previously implied, will yield formula 3.17(1) for radius $a \ll l$.

4.4 Another special application of 4.1(2) is that of determining the mutual impedance of collinear antennas. As in the case of self impedances, let $r = a$. Then, for center to center spacing h, and for

$$l_1 \gg a, \; l_2 \gg a, \; h \gg a,$$

$$r_i = \sqrt{a^2 + y_i^2} \doteq y_i + \frac{a^2}{2y_i}. \tag{1}$$

Thus, for $i \neq 1$,

$$L_{i2} \doteq 2y_i, \; L_{i1} \doteq \frac{a^2}{2y_i}, \tag{2}$$

and for $i = 1$,

$$\begin{aligned} L_{11} &= \sqrt{a^2 + y_1^2} - y_1, \\ L_{12} &= \sqrt{a^2 + y_1^2} + y_1. \end{aligned} \tag{3}$$

Thus, for $i \neq 1$,

$$\mathrm{Ci}\,kL_{i1} \doteq C + \ln\frac{a^2}{2y_i}, \; \mathrm{Si}\,kL_{i1} \doteq 0, \tag{4}$$

and one obtains from 3.9(1,2) :

$$
\begin{aligned}
R_{21}\frac{\sin kl_1 \sin kl_2}{16} = {} \\
& \sin ky_4[\mathrm{Si}\,2ky_4 - \mathrm{Si}\,2ky_2] \\
& -\cos ky_4[\mathrm{Ci}\tfrac{ka^2}{2y_2} - \mathrm{Ci}\tfrac{ka^2}{2y_4} - \mathrm{Ci}\,2ky_4 + \mathrm{Ci}\,2ky_2] \\
& +\sin ky_9[\mathrm{Si}\,2ky_9 - \mathrm{Si}\,2ky_8] \\
& -\cos ky_9[\mathrm{Ci}\tfrac{ka^2}{2y_8} - \mathrm{Ci}\tfrac{ka^2}{2y_9} - \mathrm{Ci}\,2ky_9 + \mathrm{Ci}\,2ky_8] \\
& -2\cos kl_2 \sin ky_7[\mathrm{Si}\,2ky_7 - \mathrm{Si}\,2kh] \\
& +2\cos kl_2 \cos ky_7[\mathrm{Ci}\tfrac{ka^2}{2h} - \mathrm{Ci}\tfrac{ka^2}{2y_7} - \mathrm{Ci}\,2ky_7 + \mathrm{Ci}\,2kh] \\
& -\sin ky_1[\mathrm{Si}\,2ky_2 - \mathrm{Si}\,2ky_1] \\
& +\cos ky_1[\mathrm{Ci}\,k(\sqrt{a^2+y_1^2} - y_1) - \mathrm{Ci}\tfrac{ka^2}{2y_2} - \mathrm{Ci}\,2ky_2 + \mathrm{Ci}\,k(\sqrt{a^2+y_1^2} + y_1)] \\
& -\sin ky_6[\mathrm{Si}\,2ky_8 - \mathrm{Si}\,2ky_6] \\
& +\cos ky_6[\mathrm{Ci}\tfrac{ka^2}{2y_6} - \mathrm{Ci}\tfrac{ka^2}{2y_8} - \mathrm{Ci}\,2ky_8 + \mathrm{Ci}\,2ky_6] \\
& +2\cos kl_2 \sin ky_3[\mathrm{Si}\,2kh - \mathrm{Si}\,2ky_3] \\
& -2\cos kl_2 \cos ky_3[\mathrm{Ci}\tfrac{ka^2}{2y_3} - \mathrm{Ci}\tfrac{ka^2}{2h} - \mathrm{Ci}\,2kh + \mathrm{Ci}\,2ky_3]
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
X_{21}\frac{\sin kl_1 \sin kl_2}{16} = {} \\
& \sin ky_4[\mathrm{Ci}\tfrac{ka^2}{2y_2} - \mathrm{Ci}\tfrac{ka^2}{2y_4} + \mathrm{Ci}\,2ky_4 - \mathrm{Ci}\,2ky_2] \\
& +\cos ky_4[\mathrm{Si}\,2ky_2 - \mathrm{Si}\,2ky_4] \\
& +\sin ky_9[\mathrm{Ci}\tfrac{ka^2}{2y_8} - \mathrm{Ci}\tfrac{ka^2}{2y_9} + \mathrm{Ci}\,2ky_9 - \mathrm{Ci}\,2ky_8] \\
& +\cos ky_9[\mathrm{Si}\,2ky_8 - \mathrm{Si}\,2ky_9] \\
& -2\cos kl_2 \sin ky_7[\mathrm{Ci}\tfrac{ka^2}{2h} - \mathrm{Ci}\tfrac{ka^2}{2y_7} + \mathrm{Ci}\,2ky_7 - \mathrm{Ci}\,2kh] \\
& -2\cos kl_2 \cos ky_7[\mathrm{Si}\,2kh - \mathrm{Si}\,2ky_7] \\
& -\sin ky_1[\mathrm{Ci}\,k(\sqrt{a^2+y_1^2} - y_1) - \mathrm{Ci}\tfrac{ka^2}{2y_2} + \mathrm{Ci}\,2ky_2 - \mathrm{Ci}\,k(\sqrt{a^2+y_1^2} + y_1)] \\
& -\cos ky_1[\mathrm{Si}\,2ky_1 - \mathrm{Si}\,2ky_2] \\
& -\sin ky_6[\mathrm{Ci}\tfrac{ka^2}{2y_6} - \mathrm{Ci}\tfrac{ka^2}{2y_8} + \mathrm{Ci}\,2ky_8 - \mathrm{Ci}\,2ky_6] \\
& -\cos ky_6[\mathrm{Si}\,2ky_6 - \mathrm{Si}\,2ky_8] \\
& +2\cos kl_2 \sin ky_3[\mathrm{Ci}\tfrac{ka^2}{2y_3} - \mathrm{Ci}\tfrac{ka^2}{2h} + \mathrm{Ci}\,2kh - \mathrm{Ci}\,2k(h-l_1)] \\
& +2\cos kl_2 \cos ky_3[\mathrm{Si}\,2ky_3 - \mathrm{Si}\,2kh]
\end{aligned}
$$

For two collinear antennas of equal length, each of radius $\underline{a}$, and with the spacings between adjacent ends equal to either zero or $\underline{a}$, formula (6) reduces to

$$R_{21} = \frac{15}{\sin^2 kl}\Big\{[C + \ln kl - 2Ci2kl + Ci4kl]$$
$$+ 2\sin 2kl[-Si2kl + 2Si4kl - Si6kl] + 2\cos 2kl[\ln\tfrac{3}{4} - Ci2kl + 2Ci4kl - Ci6kl]$$
$$+ \sin 4kl[Si4kl - 2Si6kl + Si8kl] + \cos 4kl[\ln\tfrac{9}{8} + Ci4kl - 2Ci6kl + Ci8kl]\Big\}$$

$$(6)$$

$$X_{21} = \frac{15}{\sin^2 kl}\Big\{[2Si2kl - Si4kl] + 2\sin 2kl[\ln\tfrac{4}{3} - Ci2kl + 2Ci4kl - Ci6kl]$$
$$+ 2\cos 2kl[Si2kl - 2Si4kl + Si6kl] + \sin 4kl[\ln\tfrac{8}{9} + Ci4kl - 2Ci6kl + Ci8kl]$$
$$+ \cos 4kl[-Si4kl + 2Si6kl - Si8kl]\Big\}$$

Formula (6) is useful in computing the input impedance of resonant long wire antennas.

4.5 To arrive at a formula for the input impedance of a harmonically operated antenna, first consider the full wave antenna, that is, an antenna fed at the center of one half-wave section. It will be assumed that the currents in the two half-wave sections are numerically equal but in phase opposition.

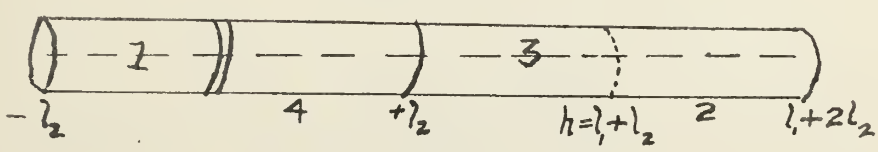


Figure 10

The integration for the fields is carried out over paths one and three, and the current integration is carried out over paths two and four. Thus, symbollically,

$$(\phi_1-\phi_3)(\phi_4-\phi_2) = \phi_1\phi_4-\phi_3\phi_4-\phi_1\phi_2+\phi_3\phi_2,$$

or

$$Z_{in} = Z_{22}+Z_{11}-2Z_{21}\ . \qquad (1)$$

For $I_2 = I_1$

$$Z_{in} = 2(Z_{22}-Z_{21})\ . \qquad (2)$$

On the other hand, if the antennas were individually fed in phase opposition with $Z_{..} = Z_{..}$, from the mesh equations one would have

$$Z_{1in} = Z_{2in} = Z_{22}-Z_{21}\ . \qquad (3)$$

Hence, one concludes that the section of the antenna between the two center points acts as a transmission line coupling the input impedance of antenna one in series with the input impedance of antenna two. Thus,

Figure 11

For operation on the third harmonic, let

$$I_1 = -I_2 = -I_3$$

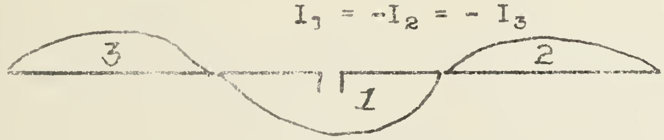


Figure 12

and

$$Z_{in} = 3Z_{11} - 4Z_{12} + 2Z_{23} \qquad (4)$$

For operation on the fourth harmonic, let

$$I_1 = -I_2 = -I_3 = I_4$$

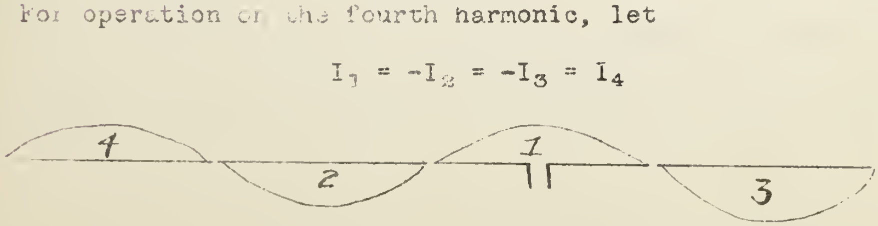


Figure 13

and

$$Z_{in} = 4Z_{11} - 6Z_{12} + 4Z_{23} - 2Z_{34} \; . \tag{5}$$

Thus, by induction, if $Z_m(\frac{\lambda}{4})$ is the mutual impedance given by formula 4.3(6) for two half wave antennas spaced

$$h = \frac{m\lambda}{4} \; ,$$

one may write the input impedance of an antenna operated at its n^{th} harmonic frequency , $n>1$, as

$$Z_{in} = nZ_{11} + 2\Sigma_1^{n-1}(-1)^{\frac{m}{2}}[n - \frac{m}{2}]Z_m(\frac{\lambda}{4}) \; , \tag{6}$$

where for $m \geq 4$,

$$Z_m(\frac{\lambda}{4}) \doteq (-1)^{\frac{m}{2}+1} \ln\frac{m^2}{(m^2-4)}{}^{30}_{15} + j0 \; . \tag{7}$$

For plotting purposes, it is also useful to have the recurrence formula

$$Z_{in}([n+1]\frac{\lambda}{2}) = Z_{in}(\frac{n\lambda}{2}) + 2\Sigma_1^{n}(-1)^{m}Z_{2m}(\frac{\lambda}{4}) \; , \tag{8}$$

where $Z_{2m}(\frac{\lambda}{4})$ is the mutual impedance of two collinear antennas each one half-wave in length and spaced

$$h = \frac{m\lambda}{2}$$

and is given by 4.4(6).

While texts and hand books list the radiation resistance of harmonic antennas, they do not show the reactive component of the input impedance. The resistive component of the above formulas yield numerical results which check very closely with values found in handbooks and texts.

For a pair of harmonically operated antennas of length $\frac{n\lambda}{2}$ each normal to the line of centers and spaced r , let $Z_n(\frac{\lambda}{4}, r)$

be the mutual impedance between two half-wave antennas with

$$h = \frac{m\lambda}{4}, \quad r_{,} = r_{,}$$

as given by 4.1(2), and one may write their mutual impedance as

$$Z_{mutual} = nZ_{21} + 2\sum_{1}^{n-1}(-1)^{\frac{m}{2}}\left(n - \frac{m}{2}\right)Z_m\left(\frac{\lambda}{4}, r_{,}\right) \quad , \qquad (9)$$

in which Z_{21} is the mutual impedance of two half-wave collinear antennas spaced a half-wave center to center.

Taking Z_{self} as Z_{in}, given in (8) above, it is then possible to solve for the input impedance of a Yagi array operated harmonically, assuming it is fed at the center of a half-wave section.

4.6 The resistive component of the self impedance of a long wire antenna is known as the radiation resistance of the antenna, provided the reference current is taken as the spatial maximum current. These values, as mentioned in 4.6 are listed in handbooks. They are customarily computed by the induced E.M.F. method, which assumes a dimensionless cross section. For checking those values against the values computed above, th following formulas are useful:

$$R_r = 240\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^n k^{2(n+1)}}{(2n+3)(2n+1)!}\binom{2n}{2r}a_{2(n-r)}a_{2r} \qquad (1)$$

with

$$a_r = \int_0^1 f(x)x^r dx \qquad (2)$$

$f(x)$ being the current distribution function. Formula (1) is for two half-wave antennas fed in phase. If the antennas are fed in phase opposition, the radiation resistance formula becomes:

$$R_r = 240\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{n-1}k^{2(n+1)}}{(2n+3)(2n+1)!}\binom{2n}{2r+1}a_{2(n-r)-1}a_{2r+1} \qquad (3)$$

If the current distribution function satisfies the following conditions,

$$f(x) \equiv g(kx), \quad g(kl) = 0$$

$$G(kx) = k\int g(kx)dx, \quad \iint f(x)dx = -f(x)$$

then the α's satisfy the following recurrence formulas:

$$\alpha_0 = \frac{1}{k}[G(kl) - G(0)]$$

$$\alpha_1 = l\alpha_0 + \frac{1}{k}G(0) - \frac{1}{k^2}g(0)$$

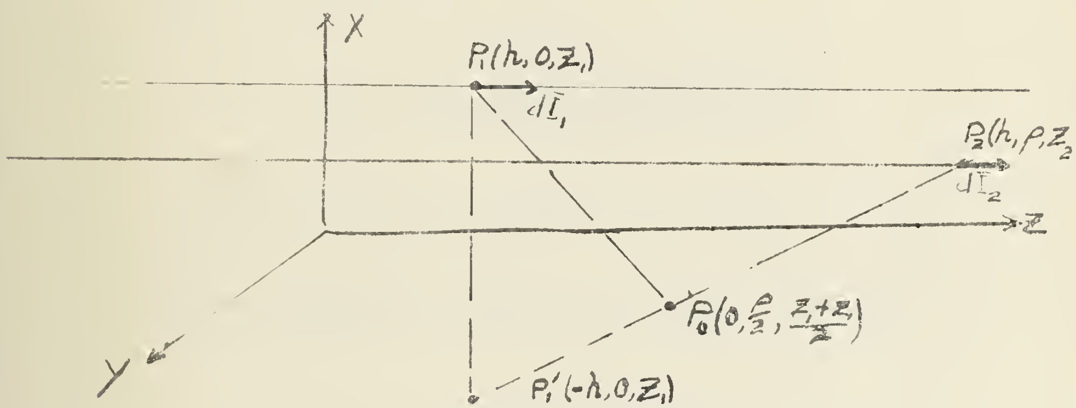
$$\alpha_r = \frac{1}{k^2}[k\, l^r G(kl) - r(r-1)\alpha_{r-2}]$$

Of course, the reactive component becomes infinite for a wire of vanishing cross section, and when it is desired to find the input impedance of a physical antenna, the radius of the wire must be taken into consideration by some method such as that of 4.5 .

V

EFFECT OF A FINITE GROUND

5.1 The preceding formulas for the mutual and self impedances will, in practice, require modification to care for the presence of ground. To this end, consider the mutual impedance between two horizontal antennas each normal to their line of centers, spaced ρ center to center, and at a height h above ground.



Let

Figure 14.

$$B = \frac{e^{-jkr}}{r} \quad \text{and} \quad d\bar{A} = \frac{I_1}{4\pi}Bdz \ .$$

Then the field $d\bar{E}$ at any point $P(x,y,z)$ due to the current element $\bar{a}_z I_1 dz_1$ is

$$d\bar{E} = \frac{30 I_1 dz_1}{jk} \bar{\bar{\Lambda}}(\bar{a}_z B) \tag{1}$$

or since

$$\bar{\bar{\Lambda}} \equiv \bar{\nabla}\bar{\nabla}\cdot + k^2$$

$$d\bar{E} = \frac{30 I_1 dz_1}{jk}\left[\bar{a}_x \frac{\partial^2}{\partial x \partial z}B + \bar{a}_y \frac{\partial^2}{\partial y \partial z}B + \bar{a}_z\left(\frac{\partial^2}{\partial z^2}+k^2\right)B\right]. \tag{2}$$

Assuming the field is reflected at $P_0(0,\frac{\rho}{2},\frac{z_1+z_2}{2})$, and applying the coefficients of reflection k_h and k_v to the horizontal and vertical

components, respectively, the reflected field at P_0 becomes

$$d\bar{E}_0\Big]_{P_0} = \frac{30 I_1 dz_1}{jk}\left\{\bar{a}_x k_v \frac{\partial^2}{\partial x \partial z} B\Big]_{P_0} + \bar{a}_y k_h \frac{\partial^2}{\partial y \partial z} B\Big]_{P_0} + \bar{a}_z k_h \left(\frac{\partial^2}{\partial z^2} + k^2\right) B\Big]_{P_0}\right\} \tag{3}$$

where

$$B\Big]_{P_0} = \frac{\exp\left[-jk\sqrt{h^2 + \frac{r^2}{4} + \frac{(z_2-z_1)^2}{4}}\right]}{\sqrt{h^2 + \frac{r^2}{4} + \frac{(z_2-z_1)^2}{4}}} = \frac{\exp\left[-j\frac{k}{2}\sqrt{(2h)^2 + r^2 + (z_2-z_1)^2}\right]}{\frac{1}{2}\sqrt{(2h)^2 + r^2 + (z_2-z_1)^2}} \tag{4}$$

Upon arrival at P_2, the reflected field $d\bar{E}_2$ is

$$d\bar{E}_2 = \frac{30 k_v I_1 dz_1}{jk}\bar{a}_x \frac{\partial^2}{\partial x \partial z} B\Big]_{P_0} + \frac{30 k_h I_1 dz_1}{jk}\left\{\bar{a}_y \frac{\partial^2}{\partial y \partial z} B\Big]_{P_0} + \bar{a}_z \left(\frac{\partial^2}{\partial z^2} + k^2\right) B\Big]_{P_0}\right\} \tag{5}$$

where

$$B\Big]_{P_0} = \frac{\exp\left[-jk\sqrt{(2h)^2 + r^2 + (z_2-z_1)^2}\right]}{\sqrt{(2h)^2 + r^2 + (z_2-z_1)^2}} . \tag{6}$$

Also, upon arrival at P_2, the direct field $d\bar{E}_1$ is

$$d\bar{E}_1 = \frac{30 I_1 dz_1}{jk}\left\{\bar{a}_x \frac{\partial^2}{\partial x \partial z} B\Big]_{P_{12}} + \bar{a}_y \frac{\partial^2}{\partial y \partial z} B\Big]_{P_{12}} + \bar{a}_z \left(\frac{\partial^2}{\partial z^2} + k^2\right) B\Big]_{P_{12}}\right\} \tag{7}$$

where

$$B\Big]_{P_{12}} = \frac{\exp\left[-jk\sqrt{r^2 + (z_2-z_1)^2}\right]}{\sqrt{r^2 + (z_2-z_1)^2}} \tag{8}$$

The resultant field $d\bar{E}$ at P_2 is

$$d\bar{E} = d\bar{E}_1 + d\bar{E}_2 \tag{9}$$

But the contribution to the mutual complex power $d^2\Psi$ due to the field from $d\bar{I}_1$ working against the current element $d\bar{I}_2$ is

$$d^2\Psi = -\frac{1}{2} d\bar{E} \cdot d\bar{I}_2^{*} = -\bar{a}_z \cdot d\bar{E} I_2^{*} dz_2 \tag{10}$$

Assuming the current distribution functions to be

$$\begin{aligned} I_1 &= I_{1m} \sin k(l_1 - |z_1|) \\ I_2 &= I_{2m} \sin k(l_2 - |z_2|) \end{aligned} \tag{11}$$

and defining the mutual impedance $\dot{Z}_{21}$ referred to the center currents by

$$d^2\dot{Z}_{21} = \frac{2d^2\Psi_{21}}{I_{1m}I'_{2m}\sin kl_1 \sin kl_2} \qquad (12)$$

one obtains

$$\dot{Z}_{21} = \frac{j30}{\sin kl_1 \sin kl_2}\int_{-l_2}^{l_2}\int_{-l_1}^{l_1} \sin k(l_1-|z_1|)\sin k(l_2-|z_2|) \left(\frac{\partial^2}{\partial z^2}+k^2\right)\left(\frac{e^{-jkr_{21}}}{r_{21}} + k_h\frac{e^{-jkr_{21}'}}{r_{21}'}\right)dz_1dz_2 \qquad (13)$$

where

$$r_{21} = \sqrt{r^2 +(z_2-z_1)^2} \quad \text{and} \quad r_{21}' = \sqrt{(2h)^2 +r^2 +(z_2-z_1)^2}$$

<u>5.2</u> It becomes evident that

$$\dot{Z}_{21} = Z_{21} +k_hZ_{21}', \qquad (1)$$

where Z_{21}' is the mutual impedance between antenna two and the image of one with the current in the image being equal to the current in one, and where Z_{21} is the free space mutual impedance of antennas one and two. Similarly,

$$\dot{Z}_{11} = Z_{11} +k_hZ_{11}'$$
$$\dot{Z}_{22} = Z_{22} +k_hZ_{22}' \qquad (2)$$

with Z_{11}' being the mutual impedance of antenna one and its image carrying the same current, and with Z_{22}' being the mutual impedance of antenna two and its image carrying the same current.

Writing the mesh equations for the system consisting of the two antennas with their images, one obtains

$$I_1(Z_{11} + k_h Z_{11}') + I_2(Z_{12} + k_h Z_{12}') = V_1$$
$$I_1(Z_{21} + k_h Z_{21}') + I_2(Z_{22} + k_h Z_{22}') = V_2 \qquad (8)$$

Similar reasonong for vertical antennas will lead to analogous results with k_h replaced by k_v .

Of course, in the above system and in any other system of equations based upon the impedance formulas derived herein, any lumped tuning impedances would necessarily be included in the self impedance constants.

VI

NUMERICAL SOLUTION OF A YAGI ARRAY

6.1 Although it has been definitely established in the literature that the current distribution in any practical antenna differs somewhat from the assumed sinusoidal distribution upon which the formulas in this paper are based, the writer believes that the formulas presented herein yield engineering solutions that are accurate within the degree of existing measurement techniques, provided the radius of the antenna is not too large and provided the lengths are in the neighborhood of a multiple of a half-wave length. Even in other cases, they should still yield the order of magnitude of the impedances, which in most instances is probably sufficient for engineering purposes.

In support of this belief, and in order to illustrate the application of the formulas to an antenna system in the presence of a finite ground, a numerical solution of a three element, horizontally oriented, Yagi array will be given, and the computed results will be compared with measured results.

It will be recalled that C.W. Harrison, Jr.(11) has presented a theory for a two element parasitic array based upon the non-assumption of a priori sinusoidal current distribution. However, this is a free space solution or, in the presence of ground, it can be applied only to half-length verticals above a perfect ground. Also, it is very difficult for an engineer to find an expression in which he may substitute his numbers and achieve an engineering solution.

Since it is desirable to clearly indicate where one substitutes the various parameters, formulas 3.17(1) and 4.2(2) will again be listed but in a slightly different form. Of course, formula 3.17(1) is not an original formula, having been used for several years. Also, formula 4.2(2) was reported derived by J. H. Tait([63]), but the derivation itself was not given. It has been derived herein as a special case of the more general formula for parallel-staggered antennas.

Thus,

$$R = [58.908 + 30(\ln kl - Ci4kl) + 30\cot^2 kl(3.118 + 3\ln kl - 4Ci2kl + Ci4kl) + 60\cot kl(Si4kl - 2Si2kl)] \tag{1}$$

$$X = [30Si4kl + 30\cot^2 kl(4Si2kl - Si4kl) + 60\cot kl(0.5772 + \ln\frac{ka^2}{l} + Ci4kl - 2Ci2kl)]$$

$$\begin{aligned}
R_{21} = {} & \{Cik(\sqrt{r^2+(l_1-l_2)^2}+l_1-l_2) + Cik(\sqrt{r^2+(l_1-l_2)^2}-l_1+l_2) \\
& -Cik(\sqrt{r^2+(l_1+l_2)^2}+l_1+l_2) - Cik(\sqrt{r^2+(l_1+l_2)^2}-l_1-l_2)\} \\
& +30\cot kl_1 \cot kl_2\{Cik(\sqrt{r^2+(l_1+l_2)^2}+l_1+l_2) + Cik(\sqrt{r^2+(l_1+l_2)^2}-l_1-l_2) \\
& +Cik(\sqrt{r^2+(l_1-l_2)^2}+l_1-l_2) + Cik(\sqrt{r^2+(l_1-l_2)^2}-l_1+l_2) \\
& -2Cik(\sqrt{r^2+l_1^2}+l_1) - 2Cik(\sqrt{r^2+l_1^2}-l_1) - 2Cik(\sqrt{r^2+l_2^2}+l_2) \\
& -2Cik(\sqrt{r^2+l_2^2}-l_2) + 4Cikr\} \\
& +30\cot kl_2\{Sik(\sqrt{r^2+(l_1+l_2)^2}+l_1+l_2) - Sik(\sqrt{r^2+(l_1+l_2)^2}-l_1-l_2) \\
& +Sik(\sqrt{r^2+(l_1-l_2)^2}+l_1-l_2) - Sik(\sqrt{r^2+(l_1-l_2)^2}-l_1+l_2) \\
& -2Sik(\sqrt{r^2+l_1^2}+l_1) + 2Sik(\sqrt{r^2+l_1^2}-l_1)\} \\
& +30\cot kl_1\{Sik(\sqrt{r^2+(l_1+l_2)^2}+l_1+l_2) - Sik(\sqrt{r^2+(l_1+l_2)^2}-l_1-l_2) \\
& -Sik(\sqrt{r^2+(l_1-l_2)^2}+l_1-l_2) + Sik(\sqrt{r^2+(l_1-l_2)^2}-l_1+l_2) \\
& -2Sik(\sqrt{r^2+l_2^2}+l_2) + 2Sik(\sqrt{r^2+l_2^2}-l_2)\}
\end{aligned} \tag{2a}$$

$$
\begin{aligned}
X_{21} = 30\Big\{ & \mathrm{Sik}\left(\sqrt{r^2+(l_1+l_2)^2}+l_1+l_2\right) +\mathrm{Sik}\left(\sqrt{r^2+(l_1+l_2)^2}-l_1-l_2\right) \\
& -\mathrm{Sik}\left(\sqrt{r^2+(l_1-l_2)^2}+l_1-l_2\right) -\mathrm{Sik}\left(\sqrt{r^2+(l_1-l_2)^2}-l_1+l_2\right)\Big\} \\
& +30\cot kl_1 \cot kl_2\Big\{2\mathrm{Sik}\left(\sqrt{r^2+l_1^2}+l_1\right) +2\mathrm{Sik}\left(\sqrt{r^2+l_1^2}-l_1\right) \\
& +2\mathrm{Sik}\left(\sqrt{r^2+l_2^2}+l_2\right) +2\mathrm{Sik}\left(\sqrt{r^2+l_2^2}-l_2\right) -\mathrm{Sik}\left(\sqrt{r^2+(l_1+l_2)^2}-l_1-l_2\right) \\
& -\mathrm{Sik}\left(\sqrt{r^2+(l_1+l_2)^2}+l_1+l_2\right) -\mathrm{Sik}\left(\sqrt{r^2+(l_1-l_2)^2}-l_1+l_2\right) \qquad \text{(2b)} \\
& -\mathrm{Sik}\left(\sqrt{r^2+(l_1-l_2)^2}+l_1-l_2\right) -4\mathrm{Sik}r\Big\} \\
& +30\cot kl_2\Big\{\mathrm{Cik}\left(\sqrt{r^2+(l_1+l_2)^2}+l_1+l_2\right) -\mathrm{Cik}\left(\sqrt{r^2+(l_1+l_2)^2}-l_1-l_2\right) \\
& +\mathrm{Cik}\left(\sqrt{r^2+(l_1-l_2)^2}+l_1-l_2\right) -\mathrm{Cik}\left(\sqrt{r^2+(l_1-l_2)^2}-l_1+l_2\right) \\
& -2\mathrm{Cik}\left(\sqrt{r^2+l_1^2}+l_1\right) +2\mathrm{Cik}\left(\sqrt{r^2+l_1^2}-l_1\right)\Big\} \\
& +30\cot kl_1\Big\{\mathrm{Cik}\left(\sqrt{r^2+(l_1+l_2)^2}+l_1+l_2\right) -\mathrm{Cik}\left(\sqrt{r^2+(l_1+l_2)^2}-l_1-l_2\right) \\
& -\mathrm{Cik}\left(\sqrt{r^2+(l_1-l_2)^2}+l_1-l_2\right) +\mathrm{Cik}\left(\sqrt{r^2+(l_1-l_2)^2}-l_1+l_2\right) \\
& -2\mathrm{Cik}\left(\sqrt{r^2+l_2^2}+l_2\right) +2\mathrm{Cik}\left(\sqrt{r^2+l_2^2}-l_2\right)\Big\}
\end{aligned}
$$

with (1) to be used for the self impedances and (2) to be used for the mutual impedances. It should be borne in mind that a is the radius of an antenna and l is its half length. Also, r is the center to center spacing, and

$$k = \frac{2\pi}{\lambda}.$$

If one does not care to use formula (1) for the self impedances, any of the other methods available in the literature could be substituted. In particular, the writer prefers the method of Schelkunoff(17) to the others. However, for consistency (1) will be used in the following example.

8.2

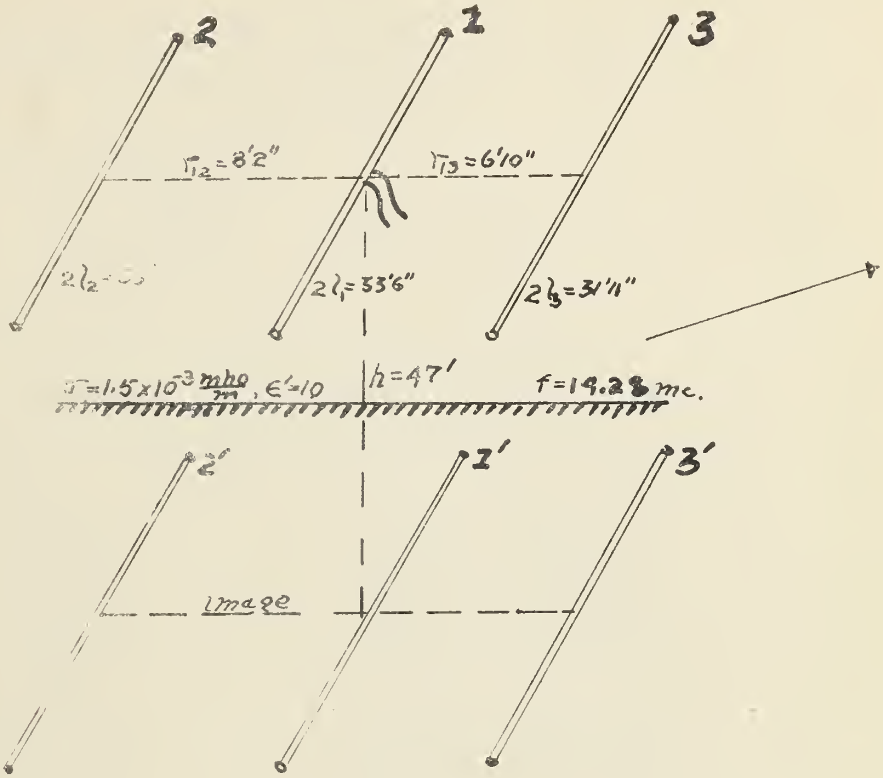


Figure 15

The system of equations for the Yagi shown above is:

$$
\begin{aligned}
(Z_{11} + k_h Z_{11}')I_1 + (Z_{12} + k_h Z_{12}')I_2 + (Z_{13} + k_h Z_{13}')I_3 &= 1 \\
(Z_{21} + K_h Z_{21}')I_1 + (Z_{22} + k_h Z_{22}')I_2 + (Z_{23} + k_h Z_{23}')I_3 &= 0 \qquad (1) \\
(Z_{31} + k_h Z_{31}')I_1 + (Z_{32} + k_h Z_{32}')I_2 + (Z_{33} + k_h Z_{33}')I_3 &= 0
\end{aligned}
$$

The well known formula for the coefficient of reflection([1,2]) for horizontally polarized waves is

$$
k_h = \frac{\cos\theta - \sqrt{\varepsilon'}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}}{\cos\theta + \sqrt{\varepsilon'}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}}, \quad \varepsilon = \varepsilon'\varepsilon_0. \qquad (2)
$$

For nearly normal incidence and for

$$\varepsilon' \gg \sin^2\theta$$

the coefficient k_h may be taken as

$$k_h = \frac{1-\sqrt{\varepsilon_c'}}{1+\sqrt{\varepsilon_c'}}, \quad \varepsilon_c' \equiv \varepsilon'\left(1 - j\frac{\sigma}{\omega\varepsilon'\varepsilon_0}\right) \quad . \tag{3}$$

Thus, for

$$\sigma = 1.5\times10^{-3}, \quad \varepsilon' = 10; \quad k_h = -0.52\angle{-4^\circ} \tag{4}$$

Converting the antenna parameters into electrical lengths and referring to tables([illegible]), the following impedances may be computed:

$Z_{11} = 67.4+j14.7$ $\qquad$ $Z_{11}'k_h = -3.83+j5.40$

$Z_{22} = 77.2+j60.8$ $\qquad$ $Z_{22}'k_h = -4.25+j5.76$

$Z_{33} = 58.9-j29.8$ $\qquad$ $Z_{33}'k_h = -3.21+j4.59$

$Z_{12} = 64.4+j1.5$ $\qquad$ $Z_{12}'k_h = -3.94+j5.80$

$Z_{13} = 57.1+j3.2$ $\qquad$ $Z_{13}'k_h = -3.41+j5.00$

$Z_{23} = 43.7-j21.4$ $\qquad$ $Z_{23}'k_h = -3.54+j6.45$

The determinant of the system becomes:

$$\Delta = \begin{vmatrix} 66.8\angle{17.7^\circ} & 61\angle{7.1^\circ} & 54.4\angle{8.9^\circ} \\ 61\angle{7.1^\circ} & 98.7\angle{42.6^\circ} & 42.9\angle{-20.4^\circ} \\ 54.4\angle{8.9} & 42.9\angle{-20.4^\circ} & 61.1\angle{-21^\circ} \end{vmatrix} \tag{6}$$

Solving for the input impedance,

$$Z_{in} = 23.2\angle{-15^\circ} = 22.4 - j6 \tag{7}$$

or the radiation resistance is

$$R_r = 22.4 \text{ ohms.}$$

The currents are

$$I_1 = 43\angle 15^\circ \text{ ma}, \quad I_2 = 11\angle 138.1^\circ \text{ ma}, \quad I_3 = 39\angle 236.8^\circ \text{ ma}, \tag{9}$$

The front to back ratio may be computed as

$$\text{db}\left(\frac{\text{front}}{\text{back}}\right) = 17.9\text{db} \tag{10}$$

The gain over a half-wave dipole in the same position is

$$g = 6.42\text{db} \tag{11}$$

and over a half-wave dipole in free space:

$$g = 5.76\text{db} \tag{12}$$

The standing wave ratio for a feed system consisting of a quarter-wave transformer whose characteristic impedance is 75 ohms working into a 300 ohm transmission line is

$$\text{SWR} = 1.4 \tag{13}$$

6.3 The front to back ratio as measured on received signals varies according to the angle of reception but is usually between 15db and [illegible]db.

The standing wave ratio was measured for several different feed systems and also computed for each system with the following results:

feed system	SWR measured	SWR computed
λ/4 75Ω — 300Ω	1.2-1.4	1.4
λ/4 72Ω — λ/4 545Ω — 470Ω	2.9-3.1	3.0
λ/4 72Ω — λ/4 440Ω — 300Ω	3.2-3.5	3.1
λ/4 72Ω — 300Ω	1.3-1.5	1.5
λ/4 52Ω — 300Ω	2.7-2.9	2.9

The exact readings for the measurements varied as the antenna was rotated but varied only approximately o.2 . This variation was due to nearby objects, principally to power lines within the vicinity. The SWR measurements were made wtih a carefully calibrated micro-match.

The above results certainly justify the belief that, in such cases, it is legitimate to assume sinusoidal current distributions for the purpose of achieving an engineering solution of an antenna problem where the radius is small compered with the length of the antenna and where the antenna lengths are in the neighborhood of a multiple of a half-wave length.

VII

INPUT IMPEDANCE OF BEVERAGE ANTENNA

7.1 The formulas for mutual impedances derived in the preceding chapters have all been for non-terminated antennas carrying only standing waves of current. It is now desirable to consider the problem of finding the input impedance of a terminated long wire antenna carrying only a travelling wave of current.

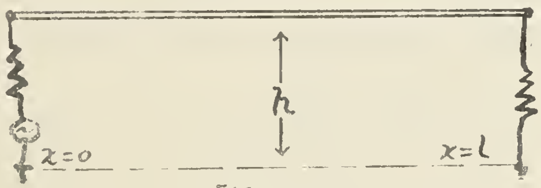


Figure 16

If a perfect ground is assumed, the problem becomes that of computing the input impedance of a parallel wire transmission line, spaced

$$\rho = 2h$$

assumed to be terminated such that standing waves are non-existent. Since the spacing is not assumed negligible by comparison with a wavelength, classical theory does not suffice. However, the radius a will be assumed such that

$$ka \ll 1, \quad a \ll l \tag{1}$$

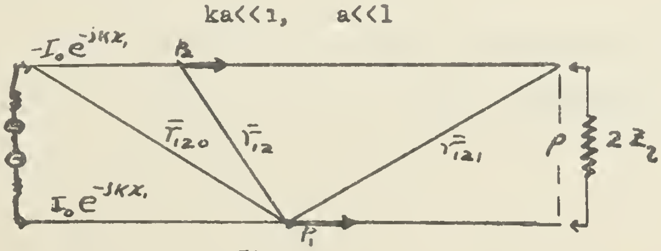


Figure 17

7.2 Following the same reasoning as in paragraph 4.4, the input impedance for the open parallel line, with infinite conductivity, becomes:

$$Z_{in} = Z_{11}+Z_{22}-Z_{12}-Z_{21}. \tag{1}$$

If it is desired to apply formulas 2.3(8-9), it should be observed that

$$Z_{21} \neq Z_{12}\,.$$

However, since

$$[f(P_1)'f(P_2)]' = f(P_2)'f(P_1) \tag{2}$$

in which the primes indicate the complex conjugate, it follows that

$$Z_{12} = \bar{Z}_{12}+\acute{Z}_{12} \tag{3}$$

and

$$Z_{21} = \bar{\bar{Z}}_{21}+\acute{Z}_{21} \tag{4}$$

where

$$\bar{\bar{Z}}_{12} = \bar{\bar{Z}}_{21};\ -\acute{Z}_{12} = \acute{Z}_{21}\ . \tag{5}$$

Then, since the self impedances are equivalent,

$$Z_{in} = 2(Z_{11}-\bar{\bar{Z}}_{21}), \tag{6}$$

which is precisely what is obtained by using formulas 2.4(12-15). In other words, the asymmetrical component of the impedances as given by the original formulas contribute nothing to the input impedance. This yields additional confirmation of the revised formulas 2.4(12-15). As a matter of fact, it was consideration of this type of problem which led to the formulation of the impedances in their final form.

7.3 Writing

$$I_1 = I_1(o)e^{-jkx_1}, \quad I_2 = I_2(o)e^{-jkx_2}, \tag{1}$$

the expression for Z_{21} becomes:

$$Z_{21} = j\frac{30}{k}\int_0^l\int_0^l \mathrm{Re}[e^{-jk(x_2-x_1)}]\bar{\bar{\Delta}}_2[\frac{e^{-jkr_{12}}}{r_{12}}d\bar{x}_1]\cdot d\bar{x}_2$$

$$= j\frac{30}{k}\int_0^l\int_0^l \cos k(x_2-x_1)[\frac{\partial^2}{\partial x_2^2}+k^2]\frac{e^{-jkr_{12}}}{r_{12}}dx_2dx_1 \quad . \tag{2}$$

Integrating by parts,

$$\frac{1}{k}\int_0^l \cos k(x_2-x_1)\frac{\partial^2}{\partial x_2^2}\frac{e^{-jkr_{12}}}{r_{12}}dx_2$$

$$= \frac{1}{k}\cos k(l-x_1)\frac{\partial}{\partial x_2}\frac{e^{-jkr_{12}}}{r_{12}}\Big]_{x_2=l} - \frac{1}{k}\cos kx_1\frac{\partial}{\partial x_2}\frac{e^{-jkr_{12}}}{r_{12}}\Big]_{x_2=o}$$

$$+\int_0^l \sin k(x_2-x_1)\frac{\partial}{\partial x_2}\frac{e^{-jkr_{12}}}{r_{12}}dx_2 \quad . \tag{3}$$

Letting

$$r_{12o} = \sqrt{r_;^2+x_1^2}\ , \quad r_{12l} = \sqrt{r_;^2+(l-x_1)^2},$$

one has

$$\frac{\partial}{\partial x_2}\frac{e^{-jkr_{12}}}{r_{12}}\Big]_{x_2=l} = -\frac{\partial}{\partial x_1}\frac{e^{-jkr_{12l}}}{r_{12l}}, \quad \frac{\partial}{\partial x_2}\frac{e^{-jkr_{12}}}{r_{12}}\Big]_{x_2=o} = -\frac{\partial}{\partial x_1}\frac{e^{-jkr_{12o}}}{r_{12o}} \tag{4}$$

Thus,

$$\frac{1}{k}\int_0^l \cos k(x_2-x_1)\frac{\partial^2}{\partial x_2^2}\frac{e^{-jkr_{12}}}{r_{12}}dx_2$$

$$= -\frac{1}{k}\cos k(l-x_1)\frac{\partial}{\partial x_1}\frac{e^{-jkr_{12l}}}{r_{12l}} + \frac{1}{k}\cos kx_1\frac{\partial}{\partial x_1}\frac{e^{-kjr_{12o}}}{r_{12o}} \tag{5}$$

$$+\int_0^l \sin k(x_2-x_1)\frac{\partial}{\partial x_2}\frac{e^{-jkr_{12}}}{r_{12}}dx_2$$

Again integrating by parts:

$$\int_0^l \sin k(x_2-x_1)\frac{\partial}{\partial x_2}\,\frac{e^{-jkr_{12}}}{r_{12}}dx_2 \tag{6}$$
$$= \sin k(l-x_1)\frac{e^{-jkr_{121}}}{r_{121}} + \sin kx_1\frac{e^{-jkr_{120}}}{r_{120}} - k\int_0^l \cos k(x_2-x_1)\frac{e^{-jkr_{12}}}{r_{12}}dx_2.$$

Substituting (6) into (5) and (5) into (2):

$$\frac{1}{j30}Z_{21} = -\frac{1}{k}\int_0^l \cos k(x_1-l)\frac{\partial}{\partial x_1}\,\frac{e^{-jkr_{121}}}{r_{121}}dx_1 + \frac{1}{k}\int_0^l \cos kx_1\frac{\partial}{\partial x_1}\,\frac{e^{-jkr_{120}}}{r_{120}}dx_1$$
$$-\frac{1}{k}\int_0^l \sin k(x_1-l)\frac{e^{-jkr_{121}}}{r_{121}}dx_1 + \int_0^l \sin kx_1\frac{e^{-jkr_{120}}}{r_{120}}\,dx_1. \tag{7}$$

Letting

$$r_o = \sqrt{r_1^2+l^2}$$

and integrating the first two terms by parts:

$$\frac{1}{j30}Z_{21} = -\frac{2e^{-jkr_1}}{kr_1} + 2\cos kl\frac{e^{-jkr_o}}{kr_o} + 2\int_0^l \sin k(l-x_1)\frac{e^{-jkr_{121}}}{r_{121}}\,dx_1$$
$$+2\int_0^l \sin kx_1\,\frac{e^{-jkr_{120}}}{r_{120}}\,dx_1\;. \tag{8}$$

Within the first integrand of (8), let

$$z = l-x_1$$

and one obtains:

$$\frac{1}{j30}Z_{21} = -\frac{2e^{-jkr_1}}{kr_1} + 2\cos kl\frac{e^{-jkr_o}}{kr_o} + 4\int_0^l \sin kx_1\frac{e^{-jkr_{120}}}{r_{120}}dx_1\;. \tag{9}$$

The last integrand of (9) is in a form to which formula 3.7(2) is applicable. Thus, breaking (9) into its real and imaginary parts and making slight trigonometric modifications: (10)

$$Z_{21} = 30\left[\frac{\sin k(r_o+l)}{kr_o} + \frac{\sin k(r_o-l)}{kr_o} - \frac{2\sin kr_1}{kr_1} + 4\mathrm{Ci}kr_1 - 2\mathrm{Ci}k(r_o+l) - 2\mathrm{Ci}k(r_o-l)\right]$$
$$+j30\left[\frac{\cos k(r_o+l)}{kr_o} + \frac{\cos k(r_o-l)}{kr_o} - \frac{2\cos kr_1}{kr_1} - 4\mathrm{Si}kr_1 + 2\mathrm{Si}k(r_o+l) + 2\mathrm{Si}k(r_o-l)\right]$$

The self impedance will be written in the same manner as in 3.16(2). Hence, for $\rho = a$, with a satisfying 7.1(1):

$$Z_{11} = 60[\ln 2kl - 0.4228 + \frac{\sin 2kl}{2kl} - Ci2kl] + j60[\frac{1+\cos 2kl}{2kl} + Si2kl - \frac{1}{ka}] \quad . \tag{11}$$

It is quite interesting to find that $Re[Z_{11}]$ agrees with the radiation resistance for the single wire as derived in standard texts(2, 5) by integrating the Poynting vector over a sphere enclosing the wire.

Formula (11) gives the free space impedance of the single wire carrying a travelling wave of current for $\sigma = \infty$. It is quite capacitive, since the dominant term is $-j\frac{60\lambda}{2\pi a}$. The input impedance of the single wire of finite length carrying a travelling wave of current above a perfect ground is then given by

$$Z = Z_{11} - Z_{12} + Z_i \tag{12}$$

with Z_{11} given by (11), with Z_{12} given by (10), and with Z_i being the internal impedance of the wire. It should be remembered that in (10), ρ is twice the height of the wire. Also, it should be emphasized that (12) is not the input impedance of the terminated wire but applies to a hypothetical case which can not be realized physically, that is, so far it has been assumed that a wire of finite length acts as if it were terminated. This anomaly will be removed subsequently.

7.4 Having found the input impedance of a hypothetical open line assumed to act as if it were terminated, the resulting formula must be interpreted physically.

First, consider formula 7.2(1) with the values of Z_{12} and Z_{11} substituted from 7.3(10) and 7.3(11), respectively, letting

$$k \doteq 0 , \quad l \gg \rho .$$

For small values of k, the internal impedance

$$Z_i = \frac{R_s l}{2\pi a}(1+j) , \quad \omega \gg 1 \tag{1}$$

must be replaced by

$$Z_i = \frac{l}{\pi a^2 \sigma} + j\frac{\omega\mu_0 l}{8\pi}, \tag{2}$$

and thus approaches the D.C.resistance.

Now, for

$$k\rho \ll 1, \quad \rho \ll l, \quad k \doteq 0,$$

one has

$$2(Z_{11} - Z_{12}) \doteq j120\left(\frac{1}{k\rho} - \frac{1}{ka}\right). \tag{3}$$

To interpret (2) physically, write

$$j\frac{120}{k}\left(\frac{1}{\rho} - \frac{1}{a}\right) = \frac{R_0}{j\omega\pi\sqrt{\mu_0\varepsilon_0}}\left(\frac{1}{a} - \frac{1}{\rho}\right)$$

or

$$2(Z_{11}-Z_{12}) \doteq \frac{4}{j\omega}\left[\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{\rho}\right)\right]. \tag{4}$$

The expression within the bracket may be recognized as the elastance of a spherical capacitor having an inner radius equivalent to the radius of the wire and an outer radius equivalent to the center to center spacing of the wires.

Since the right member of (3) contains the factor four, one such capacitor appears at each end of each wire. This end capacitance will be written

$$C_e = \frac{4\pi\varepsilon_0}{\frac{1}{a} - \frac{\cos 2kh}{2h}} \quad . \tag{5}$$

7.5 To arrive at an equivalent circuit for the terminated wire, one should review the implications of formulas 2.4(12-15) in connection with 7.2(1). The integration paths, considered as forming one continuous path, become one to four to three to two to one.

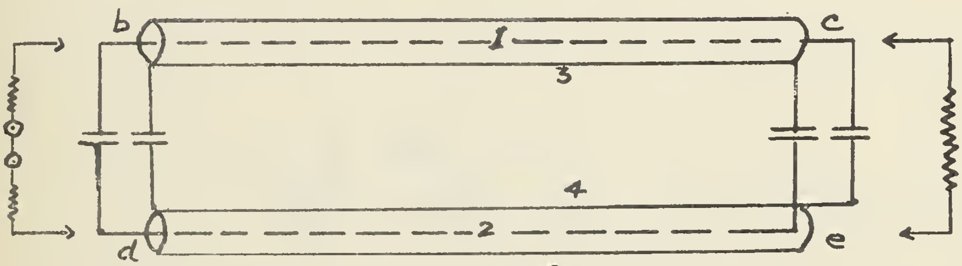


Figure 18

Thus, the end capacitors are required for completing the circuit from path to path and are therefore in series. It should be remembered that the circuit theory really calls for the field paths and the current paths to coincide. Hence, the mathematics must actually connect the paths through the capacitors. Since the applied field was assumed to be conservative, the generator may be inserted between terminals b and d, and a terminal impedance $_2Z_1=Z_0$ may be inserted between terminals c and e to complete the path and to cause the currents to remain standing waves, that is, to make the finite line act as an infinite line. The end capacitors will be in shunt with the generator and with the load, respectively.

Hence, the equivalent circuit for the two wire terminated line is:

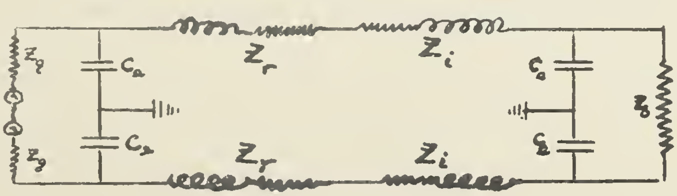


Figure 19

Since

$$\omega C_e \doteq 0$$

for

$$2h \gg a, \quad \lambda \gg a,$$

the end capacitances will be neglected. Thus, the radiation impedance Z_r may be written:

$$\begin{aligned} Z_r = 60\Big\{ &\Big[2.108 + \ln\frac{l}{\lambda} + \frac{\sin 2kl}{2kl} - Ci2kl\Big] \\ &- \Big[\cos kl\frac{\sin kr_0}{kr_0} - \frac{\sin 2kh}{2kh} + 2Ci2kh - Cik(r_0+l) - Cik(r_0-l)\Big]\Big\} \\ &+ j60\Big\{\Big[\frac{1+\cos 2kl}{2kl} + Si2kl\Big] - \Big[\cos kl\frac{\cos kr_0}{kr_0} - 2Si2kh \\ &\qquad + Sik(r_0+l) + Sik(r_0-l)\Big]\Big\} \end{aligned} \tag{1}$$

with

$$r_0 = \sqrt{l^2 + 4h^2} \quad .$$

For a long line where $l \gg r_,$,the radiation impedance reduces to

$$Z_r = 120[0.0772 + \ln kr_, - Cikr_, + \frac{\sin kr_,}{2kr_,}] + j120Sikr_, \quad , \tag{2}$$

which vanishes as

$$kr_, = 2\pi\frac{r_,}{\lambda} \doteq 0,$$

or the radiation from an ordinary open wire transmission line is negligible where line spacings are extremely small in comparison with a wave length.

7.6 The input impedance of the single wire terminated horizontal antenna placed h meters above a perfect ground may now be expressed approximately as

$$Z_{in} = Z_r + Z_i + \frac{1}{2}Z_0 \tag{3}$$

with Z_r given by 7.5(1), Z_i given by 7.4(1), and with Z_0 given by the custimary formula for the characteristic impedance of an open

wire line spaced 2h.

Formula (1) obviously can not be exact, since the attenuation of the current was ignored in the derivation of formula 7.3(10). The accuracy should be increased somewhat by writing

$$Z_0' = \sqrt{\frac{(Z_r+Z_i)+j\omega Ll}{j\omega Cl}} \tag{2}$$

with

$$L = \frac{\mu_0}{\pi}\cosh^{-1}\frac{h}{a} \tag{3}$$

$$C = \frac{\pi\varepsilon_0}{\cosh^{-1}\frac{h}{a}} \tag{4}$$

Also, the propagation constant $\alpha+j\beta$ may be written approximately as

$$\alpha+j\beta = \frac{1}{l}\sqrt{j\omega Cl[Z_r+Z_i)+j\omega Ll]} \tag{5}$$

Finally, from transmission line theory, the input impedance could be expressed as

$$Z_{in} = \tfrac{1}{2}Z_0' \quad . \tag{6}$$

7.7 For a terminated long wire antenna above a poor ground, Z_r may be approximated by applying a coefficient of reflection k_h to those terms which are due to the image antenna. Strictly speaking, since k_h as given in 6.2(2) is a function of the angle of incidence, all integrations should be repeated. However, a fair approximation should be given by writing

$$k_h = \tfrac{1}{2}(k_1+k_2) \tag{1}$$

where k_1 is computed for normal incidence, that is, for $\theta=0$, and k_2 is computed for the maximum angle of incidence, that is, for

$$\theta = \tan^{-1}\frac{l}{2h} \quad .$$

Thus, for a Beverage antenna([81]), the input impedance is given approximately by 7.6(6) with

$$Z_r = 60\Big\{[\ln 2kl - Ci2kl + \frac{\sin 2kl}{2kl} - 0.4228]$$
$$+k_h[\cos kl\frac{\sin kr_o}{kr_o} - \frac{\sin 2kh}{2kh} + 2Ci2kh - Cik(r_o+l) - Cik(r_o-l)]\Big\} \tag{2}$$
$$+j60\Big\{[\frac{1+\cos 2kl}{2kl} + Si2kl] + [\cos kl\frac{\cos kr_o}{kr_o} - 2Si2kh + Sik(r_o+l)$$
$$+Sik(r_o-l)]\Big\}$$

A more accurate but much more complicated expression for k_h would be

$$k_h = \frac{1}{\tan^{-1}\frac{l}{2h}}\int_{\theta=0}^{\theta=\tan^{-1}\frac{l}{2h}} \frac{\cos\theta - \sqrt{\varepsilon'}\sqrt{1-j\frac{\sigma}{\omega\varepsilon'\varepsilon_o}}}{\cos\theta+\sqrt{\varepsilon'}\sqrt{1-j\frac{\sigma}{\omega\varepsilon'\varepsilon_o}}}d\theta \quad . \tag{3}$$

Letting

$$b = \sqrt{\varepsilon' - j\frac{\sigma}{\omega\varepsilon'\varepsilon_o}} \tag{4}$$

the mean coefficient k_h may be written

$$k_h = 1 - \frac{2b}{\sqrt{1-b^2}\,\tan^{-1}\frac{l}{2h}}\ \ln\frac{\sqrt{l^2+4h^2}+2bh+l\sqrt{1-b^2}}{b\sqrt{l^2+4h^2}+2h} \tag{5}$$

with

$$\sqrt{1-b^2} = \sqrt{(1-\varepsilon')-j\frac{\sigma}{\omega\varepsilon_o}} \tag{6}$$

Formula (5) should yield a more accurate determination of the input impedance of the Beverage antenna when substituted into 7.7(2).

VIII

MUTUAL IMPEDANCE OF TERMINATED ANTENNAS

8.1 The integrations carried out in 7.3 will be used as a guide in in deriving an expression for the mutual impedance of two coplanar wires of length l_1 and l_2 with a lateral center to center displacement of h and a spacing of ρ .

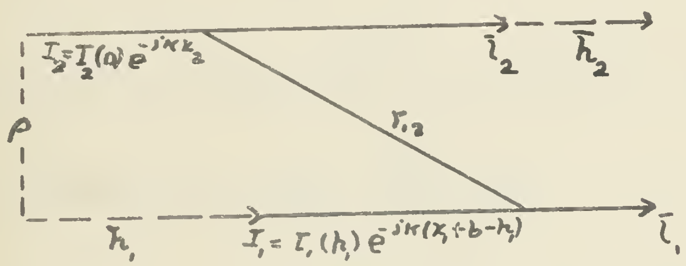


Figure 20

The currents will be assumed

$$I_1 = I_1(h_1)e^{-jk(x_1+b-h_1)}, \quad I_2 = I_2(o)e^{-jkx_2} \tag{1}$$

with $I_1(h_1)$ in phase with $I_2(o)$, and in which h_1 is defined by

$$\bar{h}_1+\bar{l}_1 = \bar{h}_2+\bar{l}_2,$$

all vectors having the same directions.

It may be readily verified that

$$\bar{h} = \frac{1}{2}(\bar{h}_1+\bar{h}_2)$$

and that

$$h_1 = h + \frac{1}{2}(l_2-l_1) .$$

8.2 The expression for Z'_{21} is

$$Z'_{21} = j\frac{30}{k}\oint_{h_1}^{l_1+h_1}\oint_{0}^{l_2}\mathrm{Re}[e^{jk(x_2-x_1-b+h_1)}]\bar{\bar{\Delta}}_2[\frac{e^{-jkr_{12}}}{r_{12}}d\bar{x}_1]\cdot d\bar{x}_2 \qquad (1)$$

$$= j\frac{30}{k}\int_{h_1}^{l_1+h_1}\int_{0}^{l_2}\cos k(x_2-x_1+h_1-b)[\frac{\partial^2}{\partial x_2^2}+k^2]\frac{e^{-jkr_{12}}}{r_{12}}dx_2dx_1 \ . \qquad (2)$$

Integrating by parts,

$$\frac{1}{k}\int_{0}^{l_2}\cos k(x_2-x_1-b+h_1)\frac{\partial^2}{\partial x_2^2}\frac{e^{-jkr_{12}}}{r_{12}}dx_2 \qquad (3)$$

$$= \frac{1}{k}\cos k(l_2-x_1-b+h_1)\frac{\partial}{\partial x_2}\frac{e^{-jkr_{12}}}{r_{12}}]_{x_2=l_2} - \frac{1}{k}\cos k(x_1+b)\frac{\partial}{\partial x_2}\frac{e^{-jkr_{12}}}{r_{12}}]_{x_2=0}$$

$$+\int_0^{l_2}\sin k(x_2-x_1-b+h_1)\frac{\partial}{\partial x_2}\frac{e^{-jkr_{12}}}{r_{12}}dx_2 \ .$$

Letting

$$r_{120} = \sqrt{\rho^2+x_1^2}, \quad r_{121} = \sqrt{\rho^2+(l_2-x_1)^2}$$

one has (4)

$$\frac{\partial}{\partial x_2}\frac{e^{-jkr_{12}}}{r_{12}}]_{x_2=l_2} = -\frac{\partial}{\partial x_1}\frac{e^{-jkr_{121}}}{r_{121}}, \quad \frac{\partial}{\partial x_2}\frac{e^{-jkr_{12}}}{r_{12}}]_{x_2=0} = -\frac{\partial}{\partial x_1}\frac{e^{-jkr_{120}}}{r_{120}}$$

Thus

$$\frac{1}{k}\int_{0}^{l_2}\cos k(x_2-x_1-b+h_1)\frac{\partial^2}{\partial x_2^2}\frac{e^{-jkr_{12}}}{r_{12}}dx_2$$

$$= -\frac{1}{k}\cos k(l_2-x_1-b+h_1)\frac{\partial}{\partial x_1}\frac{e^{-jkr_{121}}}{r_{121}} + \frac{1}{k}\cos k(x_1+b-h_1)\frac{\partial}{\partial x_1}\frac{e^{-jkr_{120}}}{r_{120}} \qquad (5)$$

$$+\int_0^{l_2}\sin k(x_2-x_1-b+h_1)\frac{\partial}{\partial x_2}\frac{e^{-jkr_{12}}}{r_{12}}dx_2 \ .$$

Again integrating by parts

$$\int_0^{l_2} \sin k(x_2-x_1+h_1-b)\frac{\partial}{\partial x_2}\,\frac{e^{-jkr_{12}}}{r_{12}}dx_2$$

$$= \sin k(l_2-x_1-b+h_1)\frac{e^{-jkr_{121}}}{r_{121}} + \sin k(x_1+b-h_1)\frac{e^{-jkr_{120}}}{r_{120}}$$

$$- \int_0^{l_2} \cos k(x_2-x_1-b+h_1)\frac{e^{-jkr_{12}}}{r_{12}}dx_2 \,. \tag{6}$$

Substituting (6) into (5) and (5) into (2) :

$$\frac{1}{j30}\,Z'_{21} =$$

$$-\frac{1}{k}\int_{h_1}^{l_1+h_1} \cos k(x_1+b-h_1-l_2)\frac{\partial}{\partial x_1}\frac{e^{-jkr_{121}}}{r_{121}}dx_1 + \int_{h_1}^{l_1+h_1} \cos k(x_1+b-h_1)\frac{\partial}{\partial x_1}\frac{e^{-jkr_{120}}}{r_{120}}dx_1$$

$$- \int_{h_1}^{h_1+l_1} \sin k(x_1+b-h_1-l_2)\frac{e^{-jkr_{121}}}{r_{121}}dx_1 + \int_{h_1}^{l_1+h_1} \sin k(x_1+b-h_1)\frac{e^{-jkr_{120}}}{r_{120}}dx_1 \tag{7}$$

Letting

$$r_0 = \sqrt{r_,^2+(l_2-h_1)^2}\,, \qquad r_1 = \sqrt{r_,^2+(l_2-l_1-h_1)^2}$$

$$r_2 = \sqrt{r_,^2+(l_1+h_1)^2}\,, \qquad r_3 = \sqrt{r_,^2+h_1^2}$$

and integrating the first two terms by parts

$$\frac{1}{j30}Z'_{21} = -\cos kb\frac{e^{-jkr_3}}{kr_3} - \cos k(l_1-l_2+b)\frac{e^{-jkr_1}}{kr_1} + \cos k(l_1+b)\frac{e^{-jkr_2}}{kr_2} + \cos k(b-l_2)\frac{e^{-jkr_0}}{kr_0} \tag{8}$$

$$+2\int_{h_1}^{l_1+h_1} \sin k(l_2+h_1-b_1-x_1)\frac{e^{-jkr_{121}}}{r_{121}}dx_1 + 2\int_{h_1}^{l_1+h_1} \sin k(x_1-h_1+b)\frac{e^{-jkr_{120}}}{r_{120}}dx_1$$

Within the first integrand of (8), let

$$z = l_2-x_1$$

and one obtains:

$$\frac{1}{j30}Z'_{21} = -\cos kb\frac{e^{-jkr_3}}{kr_3} - \cos k(l_1-l_2+b)\frac{e^{-jkr_1}}{kr_1} + \cos k(l_1+b)\frac{e^{-jkr_2}}{kr_2} + \cos k(b-l_2)\frac{e^{-jkr_0}}{kr_0} \tag{9}$$

$$+2\int_{l_2-h_1-l_1}^{l_2-h_1} \sin k(x_1+h_1-b)\frac{e^{-jkr_{120}}}{r_{120}}dx_1 + 2\int_{h_1}^{l_1+h_1} \sin k(x_1+b-h_1)\frac{e^{-jkr_{120}}}{r_{120}}dx_1$$

The two integrands are now in a form to which formula 3.7(2) is applicable. From 3.8(13),

$$l_2-h_1-l_1 = -y_{24}, \qquad l_2-h_1 = -y_{22}$$
$$h_1 = y_{25}, \qquad l_1+h_1 = y_{27} \tag{10}$$

Thus, for the first integrand,

$$r_{01} = r'_4, \qquad r_{02} = r'_2$$

and in the second integrand,

$$r_{01} = r'_5, \qquad r_{02} = r'_7 .$$

Also,

$$r_0 = r'_2, \qquad r_1 = r'_4$$
$$r_2 = r'_7 , \qquad r_3 = r'_5$$

Using the notation of 3.8(12) and dropping the superscripts:

$$2\left[\int_{-y_{24}}^{-y_{22}} \sin k(h_1-b+x_1)\frac{e^{-jkr_{120}}}{kr_{120}}dx_1 + \int_{y_{25}}^{y_{27}} \sin k(b-h_1+x_1)\frac{e^{-jkr_{120}}}{kr_{120}}dx_1\right]$$

$$\begin{aligned} = &\Big\{-\sin k(b-h_1)[\mathrm{Ci}kL_{21}-\mathrm{Ci}kL_{41}+\mathrm{Ci}kL_{42}-\mathrm{Ci}kL_{22}-\mathrm{Ci}kL_{72}+\mathrm{Ci}kL_{52}-\mathrm{Ci}kL_{51}+\mathrm{Ci}kL_{71}] \\ &+\cos k(b-h_1)[\mathrm{Si}kL_{21}-\mathrm{Si}kL_{41}-\mathrm{Si}kL_{42}+\mathrm{Si}kL_{22}+\mathrm{Si}kL_{72}-\mathrm{Si}kL_{52}-\mathrm{Si}kL_{51}+\mathrm{Si}kL_{71}]\Big\} \\ &+j\Big\{\cos k(b-h_1)[\mathrm{Ci}kL_{21}-\mathrm{Ci}kL_{41}-\mathrm{Ci}kL_{42}+\mathrm{Ci}kL_{22}+\mathrm{Ci}kL_{72}-\mathrm{Ci}kL_{52}-\mathrm{Ci}kL_{51}+\mathrm{Ci}kL_{71}] \\ &-\sin k(b-h_1)[\mathrm{Si}kL_{21}-\mathrm{Si}kL_{41}+\mathrm{Si}kL_{42}-\mathrm{Si}kL_{22}-\mathrm{Si}kL_{72}+\mathrm{Si}kL_{52}-\mathrm{Si}kL_{51}+\mathrm{Si}kL_{71}]\Big\} \end{aligned} \tag{14}$$

Writing y_{2i} as a function of h_1, that is

$$y_{2i} \equiv y_{2i}(h_1)$$

and defining y_i by

$$y_i \equiv y_{2i}(b)$$

the mutual impedance Z^1_{21} may be written as

$$
\begin{aligned}
Z'_{21} = 30\Big\{&\Big[\cos ky_2 \frac{\sin kr'_2}{kr'_2} + \cos ky_7 \frac{\sin kr'_7}{kr'_7} - \cos ky_4 \frac{\sin kr'_4}{kr'_4} - \cos ky_5 \frac{\sin kr'_5}{kr'_5} \\
&- \cos k(b-h_1)\,[\mathrm{Ci}kL_{21} - \mathrm{Ci}kL_{41} - \mathrm{Ci}kL_{42} + \mathrm{Ci}kL_{22} + \mathrm{Ci}kL_{72} - \mathrm{Ci}kL_{52} - \mathrm{Ci}kL_{51} + \mathrm{Ci}kL_{71}] \\
&+ \sin k(b-h_1)\,[\mathrm{Si}kL_{21} - \mathrm{Si}kL_{41} + \mathrm{Si}kL_{42} - \mathrm{Si}kL_{22} - \mathrm{Si}kL_{72} + \mathrm{Si}kL_{52} - \mathrm{Si}kL_{51} + \mathrm{Si}kL_{71}]\Big\} \\
+ j30\Big\{&\cos ky_2 \frac{\cos kr'_2}{kr'_2} + \cos ky_7 \frac{\cos kr'_7}{kr'_7} - \cos ky_4 \frac{\cos kr'_4}{kr'_4} - \cos ky_5 \frac{\cos kr'_5}{kr'_5} \\
&- \sin k(b-h_1)\,[\mathrm{Ci}kL_{21} - \mathrm{Ci}kL_{41} + \mathrm{Ci}kL_{42} - \mathrm{Ci}kL_{22} + \mathrm{Ci}kL_{72} - \mathrm{Ci}kL_{52} + \mathrm{Ci}kL_{51} - \mathrm{Ci}kL_{71}] \\
&+ \cos k(b-h_1)\,[\mathrm{Si}kL_{21} - \mathrm{Si}kL_{41} - \mathrm{Si}kL_{42} + \mathrm{Si}kL_{22} + \mathrm{Si}kL_{72} - \mathrm{Si}kL_{52} - \mathrm{Si}kL_{51} + \mathrm{Si}kL_{71}]\Big\}
\end{aligned}
\tag{15}
$$

As in 7.5, since the currents are given a terminal path in which to flow and are not forced to flow from the ends of the wires as displacement currents, the terms in (15) which yield the reactance of the end capacitors will be dropped. However, it is usually much safer not to drop the terms until the physical setup is analyzed. Thus, also dropping the super comma on the r's :

$$
\begin{aligned}
Z_{21} = 30\Big\{&\Big[\cos ky_2 \frac{\sin kr_2}{kr_2} + \cos ky_7 \frac{\sin kr_7}{kr_7} - \cos ky_4 \frac{\sin kr_4}{kr_4} - \cos ky_5 \frac{\sin kr_5}{kr_5}\Big] \\
&- \cos k(b-h_1)\,[\mathrm{Ci}kL_{21} - \mathrm{Ci}kL_{41} - \mathrm{Ci}kL_{42} + \mathrm{Ci}kL_{22} + \mathrm{Ci}kL_{72} - \mathrm{Ci}kL_{52} - \mathrm{Ci}kL_{51} + \mathrm{Ci}kL_{71}] \\
&+ \sin k(b-h_1)\,[\mathrm{Si}kL_{21} - \mathrm{Si}kL_{41} + \mathrm{Si}kL_{42} - \mathrm{Si}kL_{22} - \mathrm{Si}kL_{72} + \mathrm{Si}kL_{52} - \mathrm{Si}kL_{51} + \mathrm{Si}kL_{71}]\Big\} \\
+ j30\Big\{&\Big[\cos ky_2 \frac{\cos kr_2}{kr_2} + \cos ky_7 \frac{\cos kr_7}{kr_7}\Big] \\
&- \sin k(b-h_1)\,[\mathrm{Ci}kL_{21} - \mathrm{Ci}kL_{41} + \mathrm{Ci}kL_{42} - \mathrm{Ci}kL_{22} + \mathrm{Ci}kL_{72} - \mathrm{Ci}kL_{52} + \mathrm{Ci}kL_{51} - \mathrm{Ci}kL_{71}] \\
&+ \cos k(b-h_1)\,[\mathrm{Si}kL_{21} - \mathrm{Si}kL_{41} - \mathrm{Si}kL_{42} + \mathrm{Si}kL_{22} + \mathrm{Si}kL_{72} - \mathrm{Si}kL_{52} - \mathrm{Si}kL_{51} + \mathrm{Si}kL_{71}\Big\}
\end{aligned}
\tag{16}
$$

with the parameters being obtainable from 3.8(13).

Formula (16) is not applicable to collinear wires, as the reactive terms must be handled differently. Such wires will be considered in the following paragraph.

8.3 The previous previous results may be used for determining the radiation impedance of two parallel open wire transmission lines each being terminated in its characteristic impedance.

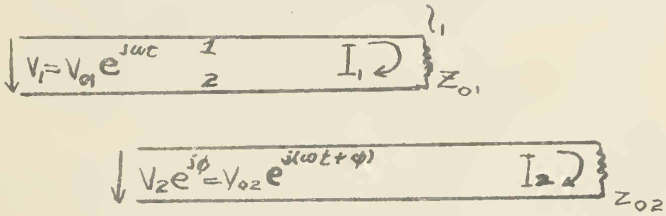


Figure 21

The self impedance of each line may be written

$$Z_{11} = Z'_{01} \ , \quad Z_{22} = Z'_{02} \tag{1}$$

with Z'_{01} and Z'_{02} given by 7.6(2).

The mutual impedances Z_{13}, Z_{14}, Z_{23}, and Z_{24} may be written by 8.2(16), in which one lets

$$kb = \phi \ . \tag{2}$$

The mesh equations are

$$\begin{aligned} Z_{11}I_1 + (Z_{13}+Z_{24}-Z_{14}-Z_{23})I_2 &= V_1 \\ (Z_{13}+Z_{24}-Z_{14}-Z_{23})I_1 + Z_{22}I_2 &= V_2 e^{j\phi} \end{aligned} \tag{4}$$

from which the input currents may be determined.

The radiation impedance of each line is given by

$$Z_r = Z_{in} - Z_o \ . \tag{5}$$

8.4 The mutual impedance of two collinear wires of radius $\underline{a}$ carrying unattenuated travelling waves may be obtained from 8.2(16)

by setting $r = a$ and making the customary approximations.

An interesting check upon the formula is provided in the derivation of the intrinsic impedance of a bi-directional wave antenna, that is, a center fed terminated wire.

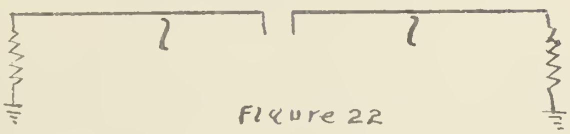


figure 22

Here,

$$\begin{aligned} &r = a, \quad l_1 = l_2 = l, \quad h_1 = l, \; b = l, \\ &y_2 = 0, \quad y_4 = l, \; y_5 = l, \quad y_7 = 2l, \\ &r_2 = a, \quad r_4 = r_5 = l + \frac{a^2}{2l}, \quad r_7 = 2l + \frac{a^2}{4l}, \end{aligned} \tag{1}$$

and

$$\begin{aligned} &L_{21} = a, \quad L_{41} = \frac{a^2}{2l} = L_{51}, \; L_{71} = \frac{a^2}{4l}, \\ &L_{22} = a, \quad L_{42} = 2l = L_{52}, \quad L_{42} = 4l. \end{aligned} \tag{2}$$

Hence

$$\begin{aligned} Z_{21} = 30\Big\{&\Big[1 + \frac{\sin 4kl}{4kl} - \frac{\sin 2kl}{kl}\Big] - \Big[C + \ln ka - 2C - 2\ln\frac{ka^2}{2l} + C \\ &+ \ln\frac{ka^2}{4l} - Ci\,2kl + C + \ln ka + Ci\,4kl - Ci\,2kl\Big]\Big\} \\ &+ j30\Big\{\Big[-\frac{1}{ka} + \frac{\cos^2 2kl}{2kl} - \frac{2\cos^2 kl}{kl}\Big] + \Big[-Si\,2kl + Si\,4kl - Si\,2kl\Big]\Big\} \end{aligned}$$

or

$$\begin{aligned} Z_{21} = 30&\Big\{1 - \frac{2\sin 2kl}{2kl} + \frac{\sin 4kl}{4kl} - C - \ln kl + 2Ci\,2kl - Ci\,4kl\Big\} \\ &+ j30\Big\{\frac{1}{ka} - \frac{1+\cos 2kl}{kl} + \frac{1+\cos 4kl}{4kl} - 2Si\,2kl + Si\,4kl\Big\} \end{aligned} \tag{3}$$

The intrinsic impedance is given by

$$Z = 2(Z_{11} + Z_{21}) \tag{4}$$

with Z_{11} given by 7.3(11).

Therefore

$$Z = 60[\ln 4kl - 0.4228 + \frac{\sin 4kl}{4kl} - Ci2kl] + j60[\frac{1+\cos 4kl}{4kl} + Si4kl - \frac{1}{ka}] \qquad (6)$$

Comparison of (6) with 7.3(11) shows that (6) is identical with the application of 7.3(11) to a wire of length 2l and thus verifies (3).

IX

MUTUAL IMPEDANCE OF OPEN AND TERMINATED WIRES

9.1 Another useful extension of the mutual impedance concept is that of the coupling between a symmetrically fed open wire antenna with an assumed sinusoidally distributed current and a terminated wire with an assumed unattenuated travelling wave of current .

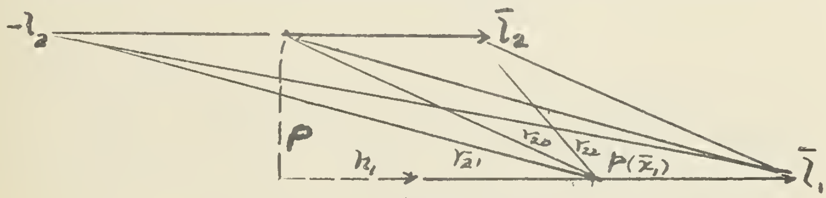


Figure 23

The directions are assumed to be the same for both wires, that is,

$$\bar{a}_1 = \bar{a}_2 \quad .$$

The currents are:

$$I_1 = I_m \sin k(l_2 - x_2) \qquad o \le x_2 \le l_2$$

$$= I_m \sin k(l_2 + x_2) \qquad -l_2 \le x_2 \le o$$

$$I_2 = I_o(h_1)e^{-jk(x_1 - h_1 + b)} \quad h_1 \le x_1 \le l_1$$

9.2 The mutual impedance becomes

$$Z_{21} = \frac{j30}{k \sin k l_2}\left[\oint_{h_1}^{h_1+l_1} \cos k(x_1 - h_1 + b)\right]$$

$$\left[\oint_{-l_2}^{o} \sin k(l_2 + x_2) + \oint_{o}^{l_2} \sin k(l_2 - x_2)\right]\bar{\bar{\Delta}}_2\left(\frac{e^{-jkr_{12}}}{r_{12}} d\bar{x}_1\right)\cdot d\bar{x}_2 \qquad (1)$$

From 3.4(8)-(12), this may be written :

$$Z_{21} = j\frac{30}{\sin kl_2}\int_{h_1}^{h_1+l_1}\cos k(x_1-h_1+b)\left[\frac{e^{-jkr_{22}}}{r_{22}}+\frac{e^{-jkr_{21}}}{r_{21}}-2\cos kl_2\frac{e^{-jkr_{20}}}{r_{20}}\right]dx_1 \quad (2)$$

in which

$$r_{22} = \sqrt{r_s^2+(x_1+l_2)^2}, \quad r_{21} = \sqrt{r_s^2+(x_1-l_2)^2}, \quad r_{20} = \sqrt{r_s^2+x_1^2} \quad .$$

In the first two terms, let

$$x = x_1+l_2$$

and

$$x = x_1-l_2$$

respectively. Then

$$Z_{21} = j\frac{30}{\sin kl_2}\Big[\int_{h_1+l_2}^{h_1+l_1+l_2}\cos k(b-h_1-l_2+x) + \int_{h_1-l_2}^{h_1+l_1-l_2}\cos k(b-h_1+l_2+x)$$

$$-2\cos kl_2\int_{h_1}^{h_1+l_1}\cos k(b-h_1+x_1)\Big]\frac{e^{-jkr_{20}}}{r_{20}}\,dx \quad (3)$$

which may be integrated by 3.7(4). For convenience in writing the results, refer to 3.8(13) and write

$$Z_{21} = j\frac{30}{\sin kl_2}\Big[\int_{y_{28}}^{y_{29}}\cos k(b-y_{28}+x) + \int_{y_{22}}^{y_{24}}\cos k(b-y_{22}+x)$$

$$-2\cos kl_2\int_{y_{25}}^{y_{27}}\cos k(b-y_{25}+x)\Big]\frac{e^{-jkr_{20}}}{r_{20}}\,dx \quad (4)$$

Hence

$$\begin{aligned} Z_{21}\frac{\sin kl_2}{j15} = \Big\{ & \cos k(y_{28}-b)[CikL_{92}-CikL_{82}+CikL_{81}-CikL_{91}] \\ & +\sin k(y_{28}-b)[SikL_{92}-SukL_{82}-SikL_{81}+SikL_{91}] \\ & +\cos k(y_{22}-b)[CikL_{42}-CikL_{22}+CikL_{21}-CikL_{41}] \\ & +\sin k(y_{22}-b)[SikL_{42}-SikL_{22}-SikL_{21}+SikL_{41}] \quad (5a) \\ & -2\cos kl_2\cos k(y_{25}-b)[CikL_{72}-CikL_{52}+CikL_{51}-CikL_{71}] \\ & -2\cos kl_2\sin k(y_{25}-b)[SikL_{72}-SikL_{52}-SikL_{51}+SikL_{71}]\Big\} \end{aligned}$$

[continued on next sheet]

$$
\begin{aligned}
&+j\Big\{\sin k(y_{28}-b)[CikL_{92}-CikL_{82}-CikL_{81}+CikL_{91}] \\
&\quad -\cos k(y_{28}-b)[SikL_{92}-SikL_{82}+SikL_{81}-SikL_{91}] \\
&\quad +\sin k(y_{22}-b)[CikL_{42}-CikL_{22}-CikL_{21}+CikL_{41}] \qquad (5b) \\
&\quad -\cos k(y_{22}-b)[SikL_{42}-SikL_{22}+SikL_{21}-SikL_{41}] \\
&-2\cos kl_2 \sin k(y_{25}-b)[CikL_{72}-CikL_{52}-CikL_{51}+CikL_{71}] \\
&-2\cos kl_2 \cos k(y_{25}-b)[SikL_{72}-SikL_{52}+SikL_{51}-SikL_{71}]
\end{aligned}
$$

which may be expressed as:

$$
\begin{aligned}
Z_{21} = \frac{15}{\sin kl_2}\Big\{&\sin k(y_{28}-b)[CikL_{72}-CikL_{52}-CikL_{51}+CikL_{71} \\
&\qquad -CikL_{92}+CikL_{82}+CikL_{81}-CikL_{91}] \\
&+\sin k(y_{22}-b)[CikL_{72}-CikL_{52}-CikL_{51}+CikL_{71} \\
&\qquad -CikL_{42}+CikL_{22}+CikL_{21}-CikL_{41}] \\
&+\cos k(y_{28}-b)[SikL_{72}-SikL_{52}+SikL_{51}-SikL_{71} \\
&\qquad +SikL_{92}-SikL_{82}+SikL_{81}-SikL_{91}] \\
&+\cos k(y_{22}-b)[SikL_{72}-SikL_{52}+SikL_{51}-SikL_{71} \\
&\qquad +CikL_{42}-CikL_{22}-CikL_{21}+CikL_{41}]\Big\} \qquad (6) \\
+j\frac{15}{\sin kl_2}\Big\{&\cos k(y_{28}-b)[CikL_{92}-CikL_{82}+CikL_{81}-CikL_{91} \\
&\qquad -CikL_{72}+CikL_{52}-CikL_{51}+CikL_{71}] \\
&+\cos k(y_{22}-b)[CikL_{42}-CikL_{22}+CikL_{21}-CikL_{41} \\
&\qquad -CikL_{72}+CikL_{52}-CikL_{51}+CikL_{71}] \\
&+\sin k(y_{28}-b)[SikL_{92}-SikL_{82}-SikL_{81}+SikL_{91} \\
&\qquad -SikL_{72}+SikL_{52}+SikL_{51}-SikL_{71}] \\
&+\sin k(y_{22}-b)[SikL_{42}-SikL_{22}-SikL_{21}+SikL_{41} \\
&\qquad -SikL_{72}+SikL_{52}+SikL_{51}-SikL_{71}]\Big\}
\end{aligned}
$$

the parameters being determinable from 3.8(13).

9.3 Formula 9.2(e) could be advantageously applied in computing the input impedance of driven arrays in which sections of the transmission line run parallel with the driven elements, such as in the following diagram:

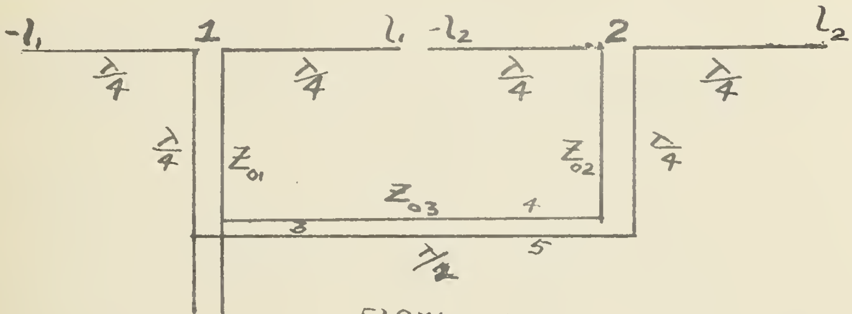


Figure 24

As a first approximation, applying classical transmission line theory for an unattenuated line, if the voltage at the junction of the lines is V_s, the input currents of the antennas will be

$$I_1 = \frac{V_s}{jZ_{01}}, \quad I_2 = \frac{-V_s}{-jZ_{02}}. \tag{1}$$

Also, although some coupling certainly exists between the antennas and the lines normal to them, to a first approximation such coupling will be assumed negligible.

Now, assuming an approximate match at the terminus of line three formed by wires four and five, the input current to line three is approximately

$$I_3 = \frac{V_s}{Z_{03}}. \tag{2}$$

Thus, for the mutual impedance of antenna one and line three:

$$Z_{13} = (Z_{15}+Z_{14}) \tag{3}$$

in which, for Z_{15}

$$b = -(\frac{\lambda}{4}+\frac{d}{2}), \quad h_5 = o, \; l_5 = \frac{\lambda}{2}, \; l_1 = \frac{\lambda}{4} \tag{4}$$

and for Z_{14},

$$b = +(\frac{\lambda}{4}-\frac{d}{2}), \quad h_4 = o, \; l_4 = \frac{\lambda}{2}, \; l_1 = \frac{\lambda}{4} \, , \tag{5}$$

in which <u>d</u> is the spacing of line three.

Similarly, for the mutual impedance of antenna two and line three,

$$Z_{23} = (Z_{25}+Z_{24}) \tag{6}$$

where for Z_{25},

$$b = +(\frac{\lambda}{4}+\frac{d}{2}), \; h_5 = -\frac{\lambda}{2}, \; l_5 = \frac{\lambda}{2}, \; l_2 = \frac{\lambda}{4} \tag{7}$$

and for Z_{24},

$$b = -(\frac{\lambda}{4}-\frac{d}{2}), \; h_5 = -\,\frac{\lambda}{2}, \; l_5 = \frac{\lambda}{2}, \; l_2 = \frac{\lambda}{4} \, , \tag{8}$$

or

$$Z_{13} = -Z_{23} \, . \tag{9}$$

The mesh equations are:

$$\begin{aligned} Z_{11}I_1 + Z_{12}I_2 + Z_{13}I_3 &= V_1 \\ Z_{21}I_1 + Z_{22}I_2 + Z_{23}I_3 &= V_2 \\ Z_{31}I_1 + Z_{32}I_2 + Z_{33}I_3 &= V_3 \end{aligned} \tag{10}$$

in which

$$Z_{33} = Z_{03}$$

and in which the various mutuals may be computed by the appropriate formulas herein.

To eliminate V_1 and V_2,, recall that Z_{02} was selected such that the input impedance of antenna two was matched to line three, that is,

$$V_2 = I_2 \frac{Z_{02}^2}{Z_{03}} \tag{11}$$

and similarly, assume Z_{01} selected to match line three to antenna one, that is,

$$V_1 = I_1 \frac{Z_{01}^2}{Z_{03}} \quad . \tag{12}$$

The input impedance to the junction is, then:

$$Z_{in} = \frac{1}{2} Z_{03} \; . \tag{13}$$

Substituting (2), (10), and (11) into (9) :

$$\begin{aligned} (Z_{11} - \frac{Z_{01}^2}{Z_{03}}) I_1 + Z_{12} I_2 + Z_{13} I_3 &= 0 \\ Z_{12} I_1 + (Z_{22} - \frac{Z_{02}^2}{Z_{03}}) I_2 + Z_{23} I_3 &= 0 \\ Z_{13} I_1 + Z_{23} I_2 \qquad\qquad\qquad &= 0 \end{aligned} \tag{15}$$

or,

$$2Z_{12}Z_{13}Z_{23} + Z_{13}^2(Z_{2in} - Z_{22}) + Z_{23}^2(Z_{1in} - Z_{11}) = 0 \tag{16}$$

Since $Z_{13} = -Z_{23}$,

$$(Z_{2in} - Z_{22} - Z_{12}) + (Z_{1in} - Z_{11} - Z_{12}) = 0 \; . \tag{17}$$

Now, $Z_{11} = Z_{22}$, and hence

$$(Z_{1in} + Z_{2in}) = 2(Z_{11} + Z_{12}). \tag{18}$$

To interpret (18), recall that if the mesh equations were set up for the two antennas alone ignoring the coupling of line three, the input impedance to each antenna would be $Z_{11}+Z_{12}$. Hence, to a first approximation, even though the operating frequence is such that line three radiates some energy, due to the physical symmetry of the system, the coupling between the antennas and the line is balanced out at the junction of the lines.

The above problem serves to illustrate how the preceding formulas might be applied. It is true that the analysis of the problem is not too rigorous, since various approximations were made which, in effect,

assumed the current distributions were unaltered by the presence of the line. However, the analysis does indicate hew one could include the coupling of the parallel portion of the open line when computing the input impedance of an antenna system. Also, the result does not imply that the coupling between the line and the antennas will always be negligible to a first approximation, since it was the physical symmetry of the system which balanced out the coupling.in this case.

X

CONCLUSION

10.1 In conclusion, it should be mentioned that although the integral equation formulation of some of the problems solved in this treatise may possibly be more rigorous mathematically, the solution of the integral equations are usually carried out by a series of approximations. In some instances, the degree of approximation may yield a more accurate solution than can be achieved by an *a priori* assumption of a current distribution based upon experimental evidence. In other instances, experimental results have checked more closely with results obtained by methods similar to those employed in this treatise. Also, the majority of the problems solved herein have not previously been considered in the same degree of generality by the integral equation method.

From an engineering point of view, both the theory and the resulting formulas presented herein are certainly worthwhile. Concerning the theory, an engineer usually prefers to consider more advanced problems as generalizations of fundamental problems with which he is already familiar. Hence, most engineers probably prefer to think of the coupling between antennas and transmission lines in terms of a generalized circuit theory. Certainly, an engineer prefers to have a formula in which he may substitute his parameters and achieve a direct solution even though he knows the solution is only a good approximation. This, in many instances, is all that is necessary.

Possibly it should be mentioned that the notion of a generalized circuit is by no means original. However, the generality to which the notion is carried is original and the formulas derived by this method, except when reduced to certain special cases, are original.

Is is believed that future applications will justify the presentation of this treatise. One such application may be that of finding the input impedance of an unidirectional rhombic antenna. If suitable approximations are made in finding a formula for the mutual impedance of the two wires of a terminated Vee antenna, then one may immediately find the input impedance of the terminated rhombic antenna with the aid of the formulas derived herein. Possibly a large number of other applications will suggest themselves in the future.

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